

On the ranks of string C-group representations for symplectic and orthogonal groups

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ABSTRACT. We determine the ranks of string C-group representations of the groups $\mathrm{PSp}(4, \mathbb{F}_q) \cong \Omega(5, \mathbb{F}_q)$, and comment on those of higher-dimensional symplectic and orthogonal groups.

1. Introduction

Abstract polytopes are incidence structures that generalize classical geometric objects such as the Platonic solids. The study of these structures (and their associated symmetry groups) continues to be a fertile area of research. Abstract *regular* polytopes and their symmetry groups are fundamentally linked by the notion of a *string C-group*, namely a group G having a distinguished generating sequence $\rho_0, \dots, \rho_{n-1}$ of involutions satisfying the following two conditions:

$$(1.1) \quad \forall 0 \leq i < j \leq n-1, \quad j-i > 1 \implies [\rho_i, \rho_j] = 1;$$

$$(1.2) \quad \forall I, J \subseteq \{0, \dots, n-1\}, \quad \langle \rho_i \mid i \in I \rangle \cap \langle \rho_j \mid j \in J \rangle = \langle \rho_k \mid k \in I \cap J \rangle.$$

The first condition—the *string property*—asserts that G is the quotient of a Coxeter group whose diagram is a string (under the conventional assumption that commuting generators are not joined by an edge). Indeed, if we label each edge in a n -node string diagram by the order of the corresponding product $\rho_i \rho_{i+1}$, and define Γ to be the Coxeter group having this diagram, then G is a *smooth* quotient of Γ . The second condition—known as the *intersection property*—is satisfied by all Coxeter groups; hence, condition (1.2) requires that the quotient G inherit this property from its parent group Γ . The integer n is the *rank* of the string C-group G .

Each string C-group G has an associated abstract regular polytope $\mathcal{P}(G)$ whose symmetry group is G . Conversely, if G is the symmetry group of some abstract regular polytope \mathcal{P} , one can build a generating sequence of involutions in G satisfying (1.1) and (1.2). Thus, the study of abstract regular polytopes is equivalent to the study of string C-groups [McS, Section 2], and in this paper we adopt the group perspective. For a group G , let $\mathrm{rk}(G)$ denote the set of integers n such that G has a string C-group representation of rank n . We prove the following:

THEOREM 1.1. *Let \mathbb{F}_q be the finite field with q elements and $G = \mathrm{PSp}(4, \mathbb{F}_q)$. Then $\mathrm{rk}(G) = \emptyset$ if $q = 3$, and $\mathrm{rk}(G) = \{3, 4, 5\}$ if $q \neq 3$.*

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Group	Reference(s)
Alt(6), Alt(7)	[Nu2]
PSL(3, \mathbb{F}_q), PSU(3, \mathbb{F}_q), PSp(4, \mathbb{F}_3)	[Nu1, Nu3, Nu4]
PSL(4, \mathbb{F}_q) (q even), PSU(4, \mathbb{F}_q) (q even)	[Nu1]
PSU(4, \mathbb{F}_3), PSU(5, \mathbb{F}_2)	M. Macaj & G. Jones
M_{11} , M_{22} , M_{23} , McL	[Ma]

FIGURE 1. Simple groups with no rank 3 string C-group representation. The exceptions of Macaj and Jones, initially overlooked by Nuzhin, were communicated via D. Leemans.

G	Restrictions	$\text{rk}(G)$	Reference(s)
Sym(n)	$n \geq 5$	$\{3, \dots, n-1\}$	[FL1]
Alt(n)	$n \geq 12$	$\{3, \dots, \lfloor \frac{n-1}{2} \rfloor\}$	[FL2]
$O^\pm(2n, \mathbb{F}_{2^e})$	$e \geq 2$	$\{3, \dots, 2n\}$	[BFL]
PSp($2n, \mathbb{F}_{2^e}$)	$e \geq 2$	$\{3, \dots, 2n+1\}$	[BFL]
PSL(4, \mathbb{F}_q)	q odd	$\supseteq \{3, 4\}$	[BL1]
PSp(4, \mathbb{F}_q)	$q \neq 3$	$\{3, 4, 5\}$	Theorem 1.1

FIGURE 2. Infinite families of simple groups (and their variants) known to have string C-group representations of rank at least 4.

Theorem 1.1 may be seen as contributing to an ongoing effort to determine the ranks of string C-group representations for families of finite simple groups.

A great deal is known about rank 3 representations: see [Co1, Co2, Nu2] for the alternating groups, [Nu1, Nu3, Nu4] for the simple groups of Lie type, and [Ma] for the sporadic simple groups. The upshot is that most finite simple groups have string C-group representations of rank 3; Figure 1 lists the exceptions.

The picture for ranks higher than 4 is far less complete. Some negative results have been proved for linear groups of low Lie rank: for example, none of PSL(3, \mathbb{F}_q), PSU(3, \mathbb{F}_q), PSL(4, \mathbb{F}_{2^e}), PSU(4, \mathbb{F}_{2^e}) have string C-group representations of any rank [BV, BL1, BFL], while $4 \in \text{rk}(\text{PSL}(2, \mathbb{F}_q))$ if, and only if, $q \in \{11, 19\}$ [LS]. Figure 2 summarizes the positive results for infinite families of simple groups and their variants.

To prove Theorem 1.1 we use the Klein correspondence to move from the 4-dimensional projective representation of PSp(4, \mathbb{F}_q) to the 5-dimensional linear representation of its isomorphic orthogonal group $\Omega(5, \mathbb{F}_q)$. This was also the approach in [BFL], where it was shown that $\text{PSp}(2n, \mathbb{F}_{2^e}) \cong \Omega(2n+1, \mathbb{F}_{2^e})$ has string C-group representations of rank $2n+1$. (Note, in odd characteristic this isomorphism holds only when $n = 2$.) That construction may be adapted in odd characteristic to generate $O(5, \mathbb{F}_q)$ as a string C-group of rank 5, but we must work harder to obtain such a representation for its simple subgroup $\Omega(5, \mathbb{F}_q)$ of index 4. Having done so, we apply a recently discovered technique [BL2] to reduce the rank of this representation and obtain the desired constructions of ranks 3 and 4.

2. Orthogonal groups and their geometries

Let \mathbb{F}_q be the field with $q = p^e$ elements, and V a d -dimensional \mathbb{F}_q -space. Although we shall be primarily interested in the case $d = 5$, we work for a while with arbitrary d . Let $\varphi: V \rightarrow \mathbb{F}_q$ be a quadratic form on V , and

$$(2.1) \quad \forall u, v \in V, \quad (u, v) = \varphi(u + v) - \varphi(u) - \varphi(v)$$

its associated symmetric bilinear form. If q is odd, φ can be recovered from $(,)$ using the equation $\varphi(v) = (v, v)/2$.

To each $U \leq V$ we associate a subspace $U^\perp = \{v \in V \mid (v, U) = 0\}$, and say V^\perp is the *radical* of V . If d is even, or d and $|\mathbb{F}_q|$ are both odd, we insist that $V^\perp = 0$. If d is odd and $|\mathbb{F}_q|$ is even, V^\perp is always nonzero; here, we insist that $V^\perp = \langle z \rangle$ with $\varphi(z) \neq 0$. A subspace U of V is *nonsingular* if the restriction of φ to U has these properties. In particular, a 1-space $\langle v \rangle$ is *nonsingular* if $\varphi(v) \neq 0$, and is otherwise *singular* (we extend this terminology to vectors).

The *orthogonal group* corresponding to φ is the group

$$(2.2) \quad O(V) = \{g \in \text{GL}(V) \mid \varphi(vg) = \varphi(v) \text{ for all } v \in V\}$$

of *isometries* of φ . Although we generally work with orthogonal groups and their underlying \mathbb{F}_q -spaces using standard (coordinate-free) linear transformation notation, it will from time to time be helpful to compute explicitly with matrices. If x is a matrix, x^{tr} denotes its transpose. Fixing an ordered basis v_1, \dots, v_d for V , we represent φ as a $d \times d$ upper-triangular matrix $\Phi := [[\varphi_{ij}]]$, where $\varphi_{ii} = \varphi(v_i)$, and $\varphi_{ij} = (v_i, v_j)$ for $1 \leq i < j \leq d$. Then $\Phi + \Phi^{\text{tr}}$ is the matrix $[[(v_i, v_j)]]$ representing $(,)$ relative to v_1, \dots, v_d , and

$$(2.3) \quad O(V) = \{g \in \text{GL}(d, \mathbb{F}_q) \mid g\Phi g^{\text{tr}} = \Phi\}.$$

Throughout this section we shall appeal to standard results from Taylor's text [Tay], to which we refer the reader for a thorough treatment of classical groups and their geometries.

2.1. Symmetries, and their geometric properties. The involutions best suited to generating string C-groups of high rank are those with ± 1 -eigenspaces of highest possible dimension. Such involutions arise from elements of $O(V)$ known as *symmetries* that are defined in terms of a nonsingular vector $u \in V$ as follows:

$$(2.4) \quad \forall v \in V, \quad \sigma_u: v \mapsto v - \frac{(u, v)}{\varphi(u)} u.$$

Observe that σ_u is the identity on the $(d - 1)$ -space u^\perp . If $|\mathbb{F}_q|$ is odd, $\sigma_u(u) = -u$, so σ_u is a *reflection* and has determinant -1 . If $|\mathbb{F}_q|$ is even, σ_u is also the identity on $V/\langle u \rangle$, and hence is a *transvection*. Note, $\sigma_u = \sigma_{\lambda u}$ for all $\lambda \in \mathbb{F}_q^\times$, so symmetries correspond to nonsingular points of the projective geometry $\mathbb{P}(V)$ and we write $\sigma_u = \sigma_{\langle u \rangle}$. For $X \subseteq V$, define

$$(2.5) \quad \Sigma(X) = \langle \sigma_u \mid u \in X \rangle.$$

We typically frame our arguments in geometric terms, and the following elementary observation is particularly useful. For nonsingular points x, y in $\mathbb{P}(V)$,

$$(2.6) \quad [\sigma_x, \sigma_y] = 1 \iff y \text{ lies on } x^\perp.$$

The *support* of $h \in \mathrm{GL}(V)$ is the subspace $[V, h] := \{v - vh \mid v \in V\}$. If $H \leq \mathrm{GL}(V)$ is generated by S , then its support is $[V, H] := \sum_{h \in S} [V, h]$.

2.2. Generating with symmetries. It is well known that unless $V = \mathbb{F}_2^4$ the orthogonal group $\mathrm{O}(V)$ is generated by its symmetries; cf. [Tay, Theorem 11.39]. If $\dim V$ is odd and $|\mathbb{F}_q|$ is even—often considered the degenerate case—then $\mathrm{O}(V) \cong \mathrm{Sp}(V/V^\perp)$ is the simple group we wish to work with. In the other cases $\mathrm{O}(V)$ has normal subgroups, which we now describe.

The map $g \mapsto \dim[V, g] \pmod{2}$ is a group homomorphism $\pi: \mathrm{O}(V) \rightarrow \mathbb{Z}_2$ having kernel of index 2. In fact, if $g = \sigma_{u_1} \dots \sigma_{u_r}$, then $\dim[V, g] \pmod{2} = r \pmod{2}$ and $\ker \pi$ is the subgroup of $\mathrm{O}(V)$ consisting of products of even numbers of symmetries. Next, the map $g \mapsto \varphi(u_1) \dots \varphi(u_r) (\mathbb{F}_q^\times)^2$ is a (well-defined) homomorphism $\theta: \mathrm{O}(V) \rightarrow \mathbb{F}_q^\times / (\mathbb{F}_q^\times)^2$, and $\theta(g)$ is called the *spinor norm* of g . For convenience, we denote the image of θ by $\{\bar{0}, \bar{1}\}$, where $\bar{0}$ is the square class, and $\bar{1}$ is the non-square class. Evidently, $\ker \theta$ has index 2 in $\mathrm{O}(V)$ if, and only if, $|\mathbb{F}_q|$ is odd, in which case

$$(2.7) \quad \ker \theta = \langle \sigma_u \mid u \in V \text{ is nonsingular, and } \varphi(u) \text{ is a square in } \mathbb{F}_q^\times \rangle.$$

Our principal focus is the subgroup

$$(2.8) \quad \Omega(V) = \begin{cases} \ker \pi \cap \ker \theta & \text{if } |\mathbb{F}_q| \text{ is odd} \\ \mathrm{O}(V) & \text{if } |\mathbb{F}_q| \text{ is even, and } \dim(V) \text{ is odd} \\ \ker \pi & \text{if } |\mathbb{F}_q| \text{ is even, and } \dim(V) \text{ is even} \end{cases}$$

of $\mathrm{O}(V)$. We restrict now to the case when $\dim V$ is odd, and consider generation of $\Omega(V)$ using symmetries.

When $|\mathbb{F}_q|$ is even, $\mathrm{O}(V) = \Omega(V)$ is generated by symmetries.

When $|\mathbb{F}_q|$ is odd, $\ker \pi = \mathrm{SO}(V)$, the subgroup of determinant 1 isometries of φ . Consider the map $\mathrm{O}(V) \rightarrow \mathbb{Z}_2 \times \{\bar{0}, \bar{1}\}$ sending $g \mapsto (g\pi, g\theta)$. The proper subgroups of the codomain give three maximal subgroups of $\mathrm{O}(V)$: $M_1 = \ker \theta$ is the preimage of $\langle (1, \bar{0}) \rangle$, M_2 is the preimage of $\langle (1, \bar{1}) \rangle$, and $\mathrm{SO}(V) = \ker \pi$ is the preimage of $\langle (0, \bar{1}) \rangle$.

Consider $-1 \in \mathrm{O}(V)$. As $\det(-1) = -1$ and $M_1 \cap M_2 = \Omega(V) \leq \mathrm{SO}(V)$, it follows that -1 belongs to precisely one of the maximal subgroups M_i . As $\det(-\sigma_u) = (-1)(-1) = 1$ for each nonsingular $u \in V$, it follows that

$$\langle -\sigma_u \mid \varphi(u) \text{ is a square} \rangle, \quad \langle -\sigma_u \mid \varphi(u) \text{ is a non-square} \rangle$$

are both subgroups of $\mathrm{SO}(V)$. Indeed, since $\dim V$ is odd,

$$(2.9) \quad \Omega(V) = \langle -\sigma_u \mid \theta(-1) = \varphi(u) (\mathbb{F}_q^\times)^2 \rangle.$$

Of course, one can simply multiply φ by a suitable scalar to ensure that $\theta(-1) = \bar{0}$, but we do not wish to place constraints on the specific φ that we work with.

For $X \subseteq V$, define subsets

$$(2.10) \quad X_\square = \langle u \in X \mid \varphi(u) \text{ is a square} \rangle \quad X_\boxtimes = \langle u \in X \mid \varphi(u) \text{ is a non-square} \rangle$$

of X . We record the following result for easy reference.

LEMMA 2.1. *If (V, φ) is a 5-dimensional orthogonal \mathbb{F}_q -space, then either*

- (i) $\theta(-1) = \bar{0}$ and $\mathrm{PSp}(4, \mathbb{F}_q) \cong \Omega(V) = \langle -\sigma_u \mid u \in V_\square \rangle$, or
- (ii) $\theta(-1) = \bar{1}$ and $\mathrm{PSp}(4, \mathbb{F}_q) \cong \Omega(V) = \langle -\sigma_u \mid u \in V_\boxtimes \rangle$.

REMARK 2.2. This trick for generating $\Omega(V)$ with elements having an eigenspace of dimension $\dim V - 1$ works whenever $\dim V$ is odd. Indeed, by extending the methods in this paper it is anticipated that *for all finite fields* \mathbb{F}_q of order at least 4, the simple group $\Omega(2n+1, \mathbb{F}_q)$ has a string C-group representation of rank $2n+1$. When $\dim V = 2n$ is even, on the other hand, $-\sigma_u$ still has determinant -1 . Thus, it seems unlikely that $\Omega^\pm(2n, \mathbb{F}_q)$ has string C-group representations of rank $2n$.

The following version of [BFL, Proposition 3.3] is more restrictive in that it applies just to nonsingular subspaces of an orthogonal space V , but it also encompasses all finite fields. The restricted version is all we need here, and the proof is simpler than that of [BFL, Proposition 3.3].

PROPOSITION 2.3. *Let U, W be nonsingular subspaces of an orthogonal \mathbb{F}_q -space V such that $U \cap W$ is nonsingular. Then $\Sigma(U) \cap \Sigma(W) = \Sigma(U \cap W)$.*

PROOF. Let $G = \mathrm{O}(V)$. For $X \subseteq V$, $\mathrm{cent}_G(X) = \{g \in G \mid \forall x \in X, xg = x\}$ is the subgroup of G fixing X pointwise. As U is a nonsingular subspace of V , the stabilizer $\mathrm{stab}_G(U) = \{g \in G \mid Ug = U\}$ factorizes as a direct product

$$\mathrm{stab}_G(U) = \mathrm{cent}_G(U) \times \mathrm{cent}_G(U^\perp).$$

Furthermore, $\mathrm{cent}_G(U^\perp)$ induces the full group $\mathrm{O}(U)$ of isometries on U ; we say $\mathrm{cent}_G(U^\perp)$ is a *natural embedding* of $\mathrm{O}(U)$ in G . In particular, $\mathrm{cent}_G(U^\perp) = \Sigma(U)$.

As W and $U \cap W$ are also nonsingular, we have $\Sigma(U \cap W) \leq \Sigma(U) \cap \Sigma(W)$. If $g \in \Sigma(U) \cap \Sigma(W) = \mathrm{cent}_G(U^\perp) \cap \mathrm{cent}_G(W^\perp)$, then g is the identity on U^\perp and W^\perp and hence on $U^\perp + W^\perp = (U \cap W)^\perp$. As $U \cap W$ is nonsingular, it follows that $\mathrm{cent}_G((U \cap W)^\perp) = \Sigma(U \cap W)$, so the result follows. \square

REMARK 2.4. Restricting the types of symmetries used to generate—working instead with the groups $\Sigma(X_\square)$ and $\Sigma(X_\boxtimes)$ —yields equivalent results, namely

$$\Sigma(U_\square) \cap \Sigma(W_\square) = \Sigma((U \cap W)_\square), \quad \text{and} \quad \Sigma(U_\boxtimes) \cap \Sigma(W_\boxtimes) = \Sigma((U \cap W)_\boxtimes).$$

3. The rank 5 construction

The construction of a string C-group representation for $\mathrm{PSP}(4, \mathbb{F}_q) \cong \Omega(5, \mathbb{F}_q)$ is similar to the one given in [BFL] to prove that $\mathrm{O}(2m+1, \mathbb{F}_{2^e}) \cong \mathrm{PSP}(2m, \mathbb{F}_{2^e})$ is a string C-group of rank $2m+1$. We could of course appeal to [BFL] for the case \mathbb{F}_{2^e} and focus just on $|\mathbb{F}_q|$ odd, but we prefer to give a (somewhat uniform) self-contained treatment for all finite fields. Before digging in, though, we record the following useful shortcut to string C-group verification.

LEMMA 3.1. *Let G be a group generated by involutions $\rho_0, \dots, \rho_{n-1}$ such that*

- (i) $\langle \rho_0, \dots, \rho_{n-2} \rangle$ and $\langle \rho_1, \dots, \rho_{n-1} \rangle$ are both string C-groups, and
- (ii) $\langle \rho_0, \dots, \rho_{n-2} \rangle \cap \langle \rho_1, \dots, \rho_{n-1} \rangle = \langle \rho_1, \dots, \rho_{n-2} \rangle$.

Then $(G; \{\rho_0, \dots, \rho_{n-1}\})$ is a string C-group representation.

PROOF. See [McS, Proposition 2E16]. \square

Let (V, φ) be an orthogonal \mathbb{F}_q -space, and u_1, \dots, u_d a basis of V consisting of nonsingular vectors. Consider a putative string C-group representation of a group generated by symmetries $\sigma_{u_1}, \dots, \sigma_{u_d}$. From the commutator relation (2.6), we see that the string condition (1.1) translates to an orthogonality condition on their associated vectors u_1, \dots, u_d , namely $(u_i, u_{i+k}) = 0$ if, and only if, $k > 1$. One

(b) holds provided α^2 is selected from the set

$$(3.5) \quad \Gamma_0(\xi) = A - \left\{ \frac{1}{2}, \frac{1}{4}, \frac{\xi^2}{4\xi^2 - 1}, \frac{2\xi^2 - \frac{1}{4}}{4\xi^2 - 1}, \frac{\xi^2 - \frac{1}{4}}{4\xi^2 - 2}, \frac{\xi^2 - \frac{3}{8}}{2\xi^2 - 1} \right\}$$

To analyze condition (c), we compute an orthogonal basis $\{w_i \mid 1 \leq i \leq 5\}$ for V . For then, $-1 = \prod_i \sigma_{w_i}$, so that $\theta(-1) = \prod_i \varphi(w_i) (\mathbb{F}_q^\times)^2$. Put $U = \langle u_1, u_3, u_5 \rangle$, so that $U^\perp = \langle -u_1 + 2u_2 - u_3, -u_3 + 2u_4 - u_5 \rangle$. As

$$\varphi(-u_1 + 2u_2 - u_3) = 4\xi^2 - 2 \quad \varphi(-u_3 + 2u_4 - u_5) = 4\alpha^2 - 2,$$

and $\xi^2, \alpha^2 \neq \frac{1}{2}$, it follows that $w_1 := -u_1 + 2u_2 - u_3$ and $-u_3 + 2u_4 - u_5$ are nonsingular. Notice $\varphi(w_1) = 4\xi^2 - 2 = 2^2(\xi^2 - \frac{1}{2})$ is a square, so $\theta(\sigma_{w_1}) = \bar{0}$. Also,

$$w_2 := -u_1 + 2u_2 + (1 - 4\xi^2)u_3 + (8\xi^2 - 4)u_4 + (2 - 4\xi^2)u_5 \in w_1^\perp \cap U^\perp,$$

so we compute

$$\varphi(w_2) = 16\alpha^2(2\xi^2 - 1)^2 - (32\xi^4 - 28\xi^2 + 6).$$

Define $m(\xi) = 16(2\xi^2 - 1)^2 \neq 0$, $b(\xi) = -(32\xi^4 - 28\xi^2 + 6)$, and

$$(3.6) \quad \Gamma_{\text{odd}}(\xi) = \{\alpha^2 \in \Gamma_0(\xi) \mid \alpha^2 m(\xi) + b(\xi) \text{ is a nonzero square}\}.$$

We have shown that $\Phi_{\text{odd}}(\alpha)$ satisfies conditions (a), (b) and (c) for any $\alpha \in \Gamma_{\text{odd}}(\xi)$. Note, we could have chosen ξ so that $\xi^2 - \frac{1}{2}$ is a non-square, in which case the condition on α in (3.6) would change to $\alpha^2 m(\xi) + b(\xi)$ a non-square. This increases the pool of scalars for a fixed \mathbb{F}_q , but our choice suffices to establish existence.

PROPOSITION 3.2. *Let $q \geq 4$, \mathbb{F}_q be the field with q elements, and $\xi \in \mathbb{F}_q$ any element having the properties described separately above for q even and q odd. Let $\Gamma(\xi)$ be the set defined in (3.3) or (3.6) for q even or odd, respectively. Suppose $\Gamma(\xi) \neq \emptyset$, and fix $\alpha \in \Gamma(\xi)$. If φ is the quadratic form represented, relative to basis $\{u_i \mid 1 \leq i \leq 5\}$, by the matrix $\Phi_{\text{even}}(\alpha)$ or $\Phi_{\text{odd}}(\alpha)$, then*

$$(\Omega(V; \varphi); \{-\sigma_{u_i} \mid 1 \leq i \leq 5\})$$

is a string C-group representation of $\Omega(V) \cong \text{PSp}(4, \mathbb{F}_q)$ of Schläfli type $[p, p, p, p]$, where $p \in \{q - 1, q + 1\}$ if q is even, and $p \in \{(q - 1)/2, (q + 1)/2\}$ if q is odd.

PROOF. We first claim that, for any such choice of scalars ξ, α :

$$(3.7) \quad \text{for } 1 \leq i < j \leq 5, \quad \langle \sigma_{u_i}, \dots, \sigma_{u_j} \rangle = \begin{cases} \Sigma(\langle u_i, \dots, u_j \rangle) & \text{if } q \text{ is even} \\ \Sigma(\langle u_i, \dots, u_j \rangle_\square) & \text{if } q \text{ is odd} \end{cases}.$$

Recall that for $\alpha \in \Gamma(\xi)$, conditions (a), (b) and (c) are satisfied for the form φ represented by matrix $\Phi_{\text{even}}(\alpha)$ or $\Phi_{\text{odd}}(\alpha)$. In particular, condition (a) ensures that (3.7) holds whenever $j = i + 1$. This is the base case of an induction on $j - i$. For q even, follow the argument in [BFL, Lemma 5.3]—although the form matrix is different, only conditions (a) and (b) matter. The q odd case is identical except that $\langle \sigma_{u_i}, \dots, \sigma_{u_j} \rangle = \Sigma(\langle u_i, \dots, u_j \rangle_\square)$ since we generate only with symmetries having square spinor norm; see Remark 2.4.

We next show, again using induction, that $\sigma_{u_1}, \dots, \sigma_{u_5}$ generates a string C-group. For $1 \leq i \leq 4$, the dihedral group $\langle \sigma_{u_i}, \sigma_{u_{i+1}} \rangle$ (with its defining generators) is clearly a string C-group. For $j - i > 1$, consider the group $\langle \sigma_{u_i}, \dots, \sigma_{u_j} \rangle$. By

induction, both $\langle \sigma_{u_i}, \dots, \sigma_{u_{j-1}} \rangle$ and $\langle \sigma_{u_{i+1}}, \dots, \sigma_{u_j} \rangle$ are string C-groups. Furthermore, by Proposition 2.3 and the claim above,

$$\begin{aligned} \langle \sigma_{u_i}, \dots, \sigma_{u_{j-1}} \rangle \cap \langle \sigma_{u_{i+1}}, \dots, \sigma_{u_j} \rangle &= \Sigma(\langle u_i, \dots, u_{j-1} \rangle) \cap \Sigma(\langle u_{i+1}, \dots, u_j \rangle) \\ &= \Sigma(\langle u_i, \dots, u_{j-1} \rangle \cap \langle u_{i+1}, \dots, u_j \rangle) \\ &= \Sigma(\langle u_{i+1}, \dots, u_{j-1} \rangle) \\ &= \langle \sigma_{u_{i+1}}, \dots, \sigma_{u_{j-1}} \rangle. \end{aligned}$$

It now follows from Lemma 3.1 that $\langle \sigma_{u_i}, \dots, \sigma_{u_j} \rangle$ is a string C-group.

Finally, using the initial claim again, together with the discussion in Section 2.2, we have $\langle \sigma_{u_1}, \dots, \sigma_{u_5} \rangle = \ker \theta$. When q is even, $O(V) = \ker \theta = \Omega(V)$, so $(\Omega(V); \{\sigma_{u_1}, \dots, \sigma_{u_5}\})$ is a string C-group representation. When q is odd, the choices of ξ and α ensure that condition (c) holds, so that $\theta(-1) = \bar{0}$. Thus, by Lemma 2.1(i), $(\Omega(V); \{-\sigma_{u_1}, \dots, -\sigma_{u_5}\})$ is a string C-group representation. \square

4. Proof of Theorem 1.1

We first limit the rank of a string C-group representation for $G = \text{PSp}(4, \mathbb{F}_q)$. Once again we approach this within the context of the isomorphic group $\Omega(V)$, where V is a 5-dimensional orthogonal space equipped with quadratic form φ .

LEMMA 4.1. *Let \mathbb{F}_q be any finite field, V a 5-dimension \mathbb{F}_q space equipped with quadratic form φ , and $n \geq 6$ an integer. If H is a subgroup of $O(V)$ generated by a sequence of involutions $\rho_0, \dots, \rho_{n-1}$ satisfying (1.1), then $\rho_0, \dots, \rho_{n-1}$ violates the intersection condition (1.2).*

PROOF. A non-central involution of $O(V)$ is one of $\pm\sigma_u$ or $\pm\sigma_u\sigma_w$ for $u, w \in V$ nonsingular and $w \in u^\perp$. Consider the non-commuting subgroups $L = \langle \rho_0, \rho_1, \rho_2 \rangle$ and $R = \langle \rho_3, \rho_4, \rho_5 \rangle$ of H (possible since $n \geq 6$). As each product $\rho_i\rho_{i+1}$ has order exceeding 2, L induces on $[V, L]$, of dimension at least 3, a group generated by non-commuting dihedral groups. Similarly R induces on $[V, R]$, of dimension at least 3, a second such group. As $\dim V = 5$, so $[V, L] \cap [V, R]$ is nontrivial. It follows that $L \cap R$ is also nontrivial, so condition (1.2) fails for the sequence $\rho_0, \dots, \rho_{n-1}$. \square

Lemma 4.1 shows that $\text{rk}(G) \subseteq \{3, 4, 5\}$. To complete the proof of Theorem 1.1 we establish the existence of a suitable rank 5 string C-group representation for $G \cong \Omega(V)$. This will show that $5 \in \text{rk}(G)$. We will then apply the following *rank reduction* technique to show that $\{3, 4\} \subseteq \text{rk}(G)$.

THEOREM 4.2. *Let $(G; \{\rho_0, \dots, \rho_{n-1}\})$ be an irreducible string C-group representation of rank $n \geq 4$. If $\rho_2\rho_3$ has odd order, then $(G; \{\rho_1, \rho_0\rho_2, \rho_3, \dots, \rho_{n-1}\})$ is a string C-group representation of rank $n - 1$.*

REMARK 4.3. This ‘Petrie like’ construction was developed in [HL] and first used as a rank reduction technique for the symmetric groups in [FL1]. The general technique, along with the criteria in Theorem 4.2, was established in the recent paper [BL2].

PROOF OF THEOREM 1.1. As the entire result can be verified by brute force for moderate values of q on a computer (see Section 5.4), for convenience we assume that $q \geq 4$ if q is even and that $q \geq 11$ if q is odd. Referring to Proposition 3.2, our first task is to establish the existence of a suitable scalar ξ , and to show that $\Gamma(\xi)$ is nonempty. We consider q even and odd separately.

Referring to (3.3) for q even, the only condition is that there are distinct scalars $\xi, \alpha \in \mathbb{F}_q^\times$ such that the matrices $h(\xi, \xi)$, $h(\alpha, \alpha)$, $h(\xi, \alpha)$ have order $q \pm 1$. For each $\xi, \alpha \in \mathbb{F}_q^\times - \{0, 1\}$, all of these matrices have order dividing $q \pm 1$. The proportion of scalars defining generators of those cyclic groups behaves as $1/(c \log \log q)$ and there are at least two such scalars in \mathbb{F}_q^\times for all $q \geq 4$.

Referring to (3.5) and (3.6) for q odd, while we halve the number of scalars we consider by only using squares, the proportion of these elements generating cyclic groups of the appropriate order is the same as in the even case. Upon restricting to $\Gamma_0(\xi)$, we discard up to 6 more scalars. Finally, consider the condition (as α^2 ranges over $\Gamma_0(\xi)$ for fixed ξ) that $\alpha^2 m(\xi) + b(\xi)$ is a square. If $b(\xi) = 0$, the condition always holds, but otherwise it will hold for roughly half of the choices of α^2 . In particular, $\Gamma_{\text{odd}}(q)$ is nonempty so long as q is large enough (one can check on the computer that indeed $q \geq 11$ is large enough).

Thus, by Proposition 3.2, for q large enough there is a string C-group representation of rank 5 for $\Omega(V)$. When q is odd, moreover, it is possible to choose scalars so that the Schläfli type of the representation consists of any combination of $(q-1)/2$ and $(q+1)/2$. In particular, observing the residue class of q modulo 4, we can arrange for this to be a sequence of odd numbers. (Note, when q is even, our construction always produces Schläfli types of odd numbers $q \pm 1$.) Hence, we can now simply apply Theorem 4.2 twice to our string C-group to obtain new representations of ranks 4 and 3. This completes the proof. \square

REMARK 4.4. Although our proof gives string C-group representations of rank 3 for $\text{PSp}(4, \mathbb{F}_q)$ via rank reduction, we remark that at least one other construction was already known. Indeed, as mentioned in the introduction, a complete determination of the simple groups of Lie type having string C-group representations of rank 3 may be extracted from the work of Nuzhin.

5. Concluding remarks

In this final section of the paper we make a number of remarks pertaining to the results of the foregoing sections.

5.1. Choosing scalars. We chose scalars $\xi, \alpha \in \mathbb{F}_q^\times$ in Section 3 and defined quadratic forms $\Phi_{\text{even}}(\alpha)$ and $\Phi_{\text{odd}}(\alpha)$ so as to make the verification of the corresponding string C-group representation as direct as possible. However, there is a lot of flexibility to choose other scalars in such a way that new (non-isomorphic) representations are obtained. For instance we do not absolutely require that all of the subspaces $\langle u_i, \dots, u_j \rangle$ be nonsingular; indeed, the construction in [BFL] does not insist on this.

5.2. Higher rank representations in higher dimensions. As we have already noted, the idea in [BFL] is to use symmetries to represent $\text{O}(d, \mathbb{F}_{2^e})$ as a string C-group of rank d . It was not fully appreciated at the time that the same technique can be used to build rank d representations for $\text{O}(d, \mathbb{F}_q)$ when $|\mathbb{F}_q|$ is odd. By restricting the scalars α_i in (3.1) to be either all squares or all non-squares, one can further generate the two maximal subgroups of $\text{O}(d, \mathbb{F}_q)$ generated by symmetries. It seems likely, then, that our trick for generating $\Omega(5, \mathbb{F}_q)$ by negating symmetries extends to $\Omega(2n+1, \mathbb{F}_q)$, so that $2n+1 \in \text{rk}(\Omega(2n+1, \mathbb{F}_q))$

for all $q \geq 4$. However, we have no intuition to offer in regard to the highest rank string C-group representation of the quasisimple groups $\Omega^\pm(2n, \mathbb{F}_q)$.

5.3. Another rank 4 construction. In an earlier draft of this paper—before the emergence of the rank reduction technique—a direct construction of a string C-group representation of rank 4 for $\Omega(V)$ was discovered that utilized, to a greater extent, geometric properties of involutions. For completeness we provide this construction, but in the interest of brevity omit the verification.

Let $q \geq 5$ be an odd prime power, and \mathbb{F}_q the field of q elements; an explicit rank 4 construction of $O(5, \mathbb{F}_{2^e})$ was given in [BFL, Theorem 6.1]. The tuple

$$\left[\begin{array}{cccc} -1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & -1 \end{array} \right], \left[\begin{array}{cccc} f(\alpha) & g(\alpha) & \cdot & \cdot \\ g(\alpha) & -f(\alpha) & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & -f(\alpha) & g(\alpha) \\ \cdot & \cdot & \cdot & g(\alpha) & f(\alpha) \end{array} \right], \left[\begin{array}{cccc} -1 & \cdot & \cdot & \cdot \\ \cdot & f(\beta) & -g(\beta) & \cdot \\ \cdot & -g(\beta) & -f(\beta) & \cdot \\ \cdot & \cdot & \cdot & -1 \end{array} \right], \left[\begin{array}{cccc} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{array} \right],$$

where α and β come from (different) restricted sets of scalars, and

$$(5.1) \quad f(\alpha) = \frac{1 - \alpha^2}{1 + \alpha^2} \quad g(\alpha) = \frac{-2\alpha}{1 + \alpha^2},$$

defines a string C-group representation for $\Omega(5, \mathbb{F}_q) \cong \text{PSp}(4, \mathbb{F}_q)$ of rank 4. It is not isomorphic to the one given in the proof of Theorem 1.1, which applied reduction to the rank 5 representation: the first, second, and fourth generators all have 2-dimensional -1 -eigenspace, whereas the earlier construction had three generators with 4-dimensional -1 -eigenspace.

5.4. Software. A software package called SGGI has been implemented by the author in the MAGMA system [BCP] and is publicly available on GitHub. Among other things the package contains the explicit constructions in this paper, including the one in Section 5.3, and functions to carry out exhaustive searches for string C-group representations. These functions can be used to verify Theorem 1.1 for small values of q , but we caution that $\text{PSp}(4, \mathbb{F}_{19})$ is nearing the limit of practicability for exhaustive searches. We note that other functions to compute with string C-groups, written by Dimitri Leemans, are distributed with MAGMA.

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References

- [BCP] Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265, DOI 10.1006/jsco.1996.0125. Computational algebra and number theory (London, 1993). MR1484478
- [BFL] P. A. Brooksbank, J. T. Ferrara, and D. Leemans, *Orthogonal groups in characteristic 2 acting on polytopes of high rank*, Discrete Comput. Geom. **63** (2020), no. 3, 656–669, DOI 10.1007/s00454-019-00083-0. MR4074338
- [BL1] P. A. Brooksbank and D. Leemans, *Polytopes of large rank for $\text{PSL}(4, \mathbb{F}_q)$* , J. Algebra **452** (2016), 390–400, DOI 10.1016/j.jalgebra.2015.11.051. MR3461073
- [BL2] Peter A. Brooksbank and Dimitri Leemans, *Rank reduction of string C-group representations*, Proc. Amer. Math. Soc. **147** (2019), no. 12, 5421–5426, DOI 10.1090/proc/14666. MR4021100

- [BV] Peter A. Brooksbank and Deborah A. Vicinsky, *Three-dimensional classical groups acting on polytopes*, *Discrete Comput. Geom.* **44** (2010), no. 3, 654–659, DOI 10.1007/s00454-009-9212-0. MR2679061
- [Co1] Marston D. E. Conder, *Generators for alternating and symmetric groups*, *J. London Math. Soc.* (2) **22** (1980), no. 1, 75–86, DOI 10.1112/jlms/s2-22.1.75. MR579811
- [Co2] Marston D. E. Conder, *More on generators for alternating and symmetric groups*, *Quart. J. Math. Oxford Ser. (2)* **32** (1981), no. 126, 137–163, DOI 10.1093/qmath/32.2.137. MR615190
- [FL1] Maria Elisa Fernandes and Dimitri Leemans, *Polytopes of high rank for the symmetric groups*, *Adv. Math.* **228** (2011), no. 6, 3207–3222, DOI 10.1016/j.aim.2011.08.006. MR2844941
- [FL2] Maria Elisa Fernandes and Dimitri Leemans, *String C-group representations of alternating groups*, *Ars Math. Contemp.* **17** (2019), no. 1, 291–310, DOI 10.26493/1855-3974.1953.c53. MR4031526
- [HL] Michael I. Hartley and Dimitri Leemans, *A new Petrie-like construction for abstract polytopes*, *J. Combin. Theory Ser. A* **115** (2008), no. 6, 997–1007, DOI 10.1016/j.jcta.2007.11.008. MR2423344
- [LS] Dimitri Leemans and Egon Schulte, *Groups of type $L_2(q)$ acting on polytopes*, *Adv. Geom.* **7** (2007), no. 4, 529–539, DOI 10.1515/ADVGEOM.2007.031. MR2360900
- [Ma] V. D. Mazurov, *On the generation of sporadic simple groups by three involutions, two of which commute* (Russian, with Russian summary), *Sibirsk. Mat. Zh.* **44** (2003), no. 1, 193–198, DOI 10.1023/A:1022028807652; English transl., *Siberian Math. J.* **44** (2003), no. 1, 160–164. MR1967616
- [McS] Peter McMullen and Egon Schulte, *Abstract regular polytopes*, *Encyclopedia of Mathematics and its Applications*, vol. 92, Cambridge University Press, Cambridge, 2002. MR1965665
- [MS] B. Monson and Egon Schulte, *Reflection groups and polytopes over finite fields. III*, *Adv. in Appl. Math.* **41** (2008), no. 1, 76–94, DOI 10.1016/j.aam.2007.07.001. MR2419764
- [Nu1] Ya. N. Nuzhin, *Generating triples of involutions of Chevalley groups over a finite field of characteristic 2* (Russian), *Algebra i Logika* **29** (1990), no. 2, 192–206, 261, DOI 10.1007/BF02001358; English transl., *Algebra and Logic* **29** (1990), no. 2, 134–143 (1991). MR1131150
- [Nu2] Ya. N. Nuzhin, *Generating triples of involutions of alternating groups* (Russian), *Mat. Zametki* **51** (1992), no. 4, 91–95, 142, DOI 10.1007/BF01250552; English transl., *Math. Notes* **51** (1992), no. 3-4, 389–392. MR1172472
- [Nu3] Ya. N. Nuzhin, *Generating triples of involutions of Lie-type groups over a finite field of odd characteristic. I* (Russian, with Russian summary), *Algebra i Logika* **36** (1997), no. 1, 77–96, 118, DOI 10.1007/BF02671953; English transl., *Algebra and Logic* **36** (1997), no. 1, 46–59. MR1454692
- [Nu4] Ya. N. Nuzhin, *Generating triples of involutions of Lie-type groups over a finite field of odd characteristic. II* (Russian, with Russian summary), *Algebra i Logika* **36** (1997), no. 4, 422–440, 479; English transl., *Algebra and Logic* **36** (1997), no. 4, 245–256. MR1601503
- [Tay] Donald E. Taylor, *The geometry of the classical groups*, *Sigma Series in Pure Mathematics*, vol. 9, Heldermann Verlag, Berlin, 1992. MR1189139

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