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ORTHOGONAL GROUPS IN CHARACTERISTIC 2 ACTING ON POLYTOPES OF HIGH RANK

P. A. BROOKSBANK, J. T. FERRARA, AND D. LEEMANS

ABSTRACT. We show that for all integers $m \geq 2$, and all integers $k \geq 2$, the orthogonal groups $O^\pm(2m, \mathbb{F}_{2^k})$ act on abstract regular polytopes of rank $2m$, and the symplectic groups $Sp(2m, \mathbb{F}_{2^k})$ act on abstract regular polytopes of rank $2m + 1$.

1. INTRODUCTION

In a series of three papers, Monson and Schulte studied families of crystallographic Coxeter groups with string diagrams [MS2, MS3, MS4]. In those works, the authors started with the standard real representations and considered reductions modulo odd primes, producing finite quotients of the string Coxeter groups represented in orthogonal spaces over prime fields \mathbb{F}_p . They showed that in many cases the images $\rho_0, \dots, \rho_{d-1}$ of the generating reflections of the string Coxeter group inherited a certain *intersection property* possessed by the parent group, namely

$$(1.1) \quad \forall I, J \subseteq \{0, \dots, d-1\} \quad \langle \rho_i : i \in I \rangle \cap \langle \rho_j : j \in J \rangle = \langle \rho_k : k \in I \cap J \rangle.$$

Any group generated by involutions satisfying condition (1.1) and the additional (string) commuting condition

$$(1.2) \quad \forall i, j \in \{0, \dots, d-1\} \quad \rho_i \rho_j \neq \rho_j \rho_i \iff |i - j| = 1$$

is called a *string C-group of rank d*. In [MS4, Section 3] the authors study the “3-infinity” string Coxeter groups, showing in [MS4, Theorem 3.1] that reductions of such groups modulo odd primes are always string C-groups. From this result, one can deduce the following:

Theorem 1.3. *Let p be an odd prime, and V an \mathbb{F}_p -space of dimension $d \geq 3$. There is a nondegenerate quadratic form φ on V and $\text{Isom}(V_\varphi)' \leq H \leq \text{Isom}(V_\varphi)$, where $\text{Isom}(V_\varphi)$ is the isometry group of φ , such that H is a string C-group of rank d generated by reflections.*

The modular reduction method employed by Monson and Schulte also works in some cases over extensions of \mathbb{F}_p , but it doesn’t apply at all to fields of characteristic 2. One key difference in characteristic 2 is that reflections no longer exist; they are replaced by natural analogues called *symmetries* (defined in Section 3). The main objective of the current paper is to prove the following supplement to Theorem 1.3:

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Theorem 1.4. *Let k be a positive integer, \mathbb{F}_{2^k} the field of 2^k elements, and V an \mathbb{F}_{2^k} -space of dimension $d \geq 3$. Then there is a quadratic form φ on V such that $\text{Isom}(V_\varphi)$ is a string C-group of rank d generated by symmetries if, and only if, $k \geq 2$. The radical of V is 0- or 1-dimensional according as $\dim V$ is even or odd.*

Interest in string C-groups stems from their intimate connection to highly symmetric incidence structures called *abstract regular polytopes*. Indeed a fundamental result in [MS1, Section 2] shows that \mathcal{P} is an abstract regular polytope if, and only if, $\text{Aut}(\mathcal{P})$ is a string C-group—the studies of abstract regular polytopes and string C-groups are equivalent. Refining the choice of φ in Theorem 1.4 we prove the following result concerning polytopes of high rank associated to classical groups.

Corollary 1.5. *For each integer $k \geq 2$, positive integer m , and $\epsilon \in \{-, +\}$, the orthogonal group $O^\epsilon(2m, \mathbb{F}_{2^k})$ acts as the group of automorphisms of an abstract regular polytope of rank $2m$, and the symplectic group $\text{Sp}(2m, \mathbb{F}_{2^k})$ acts as the group of automorphisms of an abstract regular polytope of rank $2m + 1$.*

Very few constructions of abstract regular polytopes of high rank are known. Indeed, aside from the Monson-Schulte constructions and the ones presented here to prove Corollary 1.5, the only existing constructions for arbitrary rank $r \geq 3$ are for the symmetric groups [FL, FLM1] and the alternating groups [FLM2].

2. SYMPLECTIC AND ORTHOGONAL SPACES

We now introduce the necessary background on classical groups and their geometries; the reader is referred to Taylor's book [T] for further information.

For the remainder of the paper, V will denote a vector space of dimension $d \geq 3$ over the finite field \mathbb{F}_q of $q = 2^k$ elements.

Let $(,)$ denote a symmetric bilinear form on V . For $U \leq V$, the subspace

$$U^\perp = \{v \in V : (v, U) = 0\}$$

is the *perp-space* of U , $\text{rad}(U) = U^\perp \cap U$ is the *radical* of U , and U is *nondegenerate* if $\text{rad}(U) = 0$. For $S \subseteq \text{GL}(V)$, the subspace

$$[V, S] = \langle v - vs : v \in V, s \in S \rangle$$

is the *support* of S . Let $\varphi: V \rightarrow \mathbb{F}_q$ be a quadratic form on V whose associated symmetric form is $(,)$ in the sense that

$$\forall u, v \in V \quad (u, v) = \varphi(u + v) + \varphi(u) + \varphi(v).$$

When we wish to stress that we are regarding V as an *orthogonal space* equipped with φ , rather than as a *symplectic space* equipped with $(,)$, we shall do so by writing V_φ . For $U \leq V_\varphi$, we write U_φ to denote the orthogonal space obtained by restricting φ to U . A nonzero vector $u \in V$ is *singular* if $\varphi(u) = 0$ and *nonsingular* otherwise. A subspace U of V is *totally singular* if $\varphi(u) = 0$ for all $u \in U$.

2.1. Matrices. We will find it convenient later on to compute with matrices representing symmetric and quadratic forms. Fix a basis v_1, \dots, v_d of V , and put $B := [[(v_i, v_j)]]$. Writing elements of V as row vectors relative to our fixed basis, one evaluates the bilinear form as

$$\forall u, v \in V \quad (u, v) = u B v^{\text{tr}}.$$

If $\Phi = [[\alpha_{ij}]]$, where $\alpha_{ij} = (v_i, v_j)$ if $i < j$, $\alpha_{ij} = 0$ if $i > j$, and $\alpha_{ii} = \varphi(v_i)$, then

$$\forall v \in V \quad \varphi(v) = v \Phi v^{\text{tr}},$$

and $B = \Phi + \Phi^{\text{tr}}$. If w_1, \dots, w_d is another basis of V and C is the matrix whose rows are the vectors representing the w_i relative to v_1, \dots, v_d , then $CBC^{\text{tr}} = [[(w_i, w_j)]]$ is the matrix representing $(\ , \)$ relative to w_1, \dots, w_d .

2.2. Classification of lines. We work with 2-dimensional subspaces of orthogonal spaces, which we often think of as lines in the associated projective space. For a line L in an orthogonal space V_φ , one of the following holds:

- L is *asingular*: all nonzero vectors are nonsingular.
- L is *singular*: $L = \langle e, b \rangle$ with $\varphi(e) = (e, b) = 0$ and $\varphi(b) \neq 0$.
- L is *hyperbolic*: $L = \langle e, f \rangle$ with $\varphi(e) = \varphi(f) = 0$ and $(e, f) \neq 0$.
- L is *totally singular*: all nonzero vectors are singular.

The four possibilities are ordered according to how many singular points (1-spaces) each possesses: 0, 1, 2, $q + 1$, respectively.

2.3. Isometries. Associated with symplectic and orthogonal spaces are their groups of isometries:

$$(2.1) \quad \text{Isom}(V) = \{g \in \text{GL}(V) : (ug, vg) = (u, v) \text{ for all } u, v \in V\}, \text{ and}$$

$$(2.2) \quad \text{Isom}(V_\varphi) = \{g \in \text{GL}(V) : \varphi(vg) = \varphi(v) \text{ for all } v \in V\}.$$

When V is nondegenerate, $\dim V = 2m$ is even and $\text{Isom}(V)$ is often called the *symplectic group* on V , denoted $\text{Sp}(V)$. In this case there are two isomorphism types of orthogonal group corresponding to two isometry types of quadratic form. These are distinguished by the dimension of a maximal totally singular subspace of V , which can be either m or $m - 1$. In the former case, the associated orthogonal group is denoted $\text{O}^+(V_\varphi)$, and in the latter case $\text{O}^-(V_\varphi)$.

We will also need to work with odd-dimensional (degenerate) orthogonal spaces V_φ when their associated symplectic spaces have 1-dimensional radicals. There are two cases to consider. First, if V^\perp is a singular point, then $\psi(v + V^\perp) := \varphi(v)$ is a well-defined nondegenerate quadratic form on V/V^\perp , and

$$(2.3) \quad \text{Isom}(V_\varphi) \cong O_2(\text{Isom}(V_\varphi)) \rtimes [\mathbb{F}_q^\times \times \text{O}^\pm(V/V^\perp)],$$

where $O_2(\text{Isom}(V_\varphi))$ —the largest normal 2-subgroup of $\text{Isom}(V_\varphi)$ —is an abelian group isomorphic to the additive group of the \mathbb{F}_q -space V/V^\perp . Hence, there are two isomorphism types for $\text{Isom}(V_\varphi)$, corresponding to the two isometry types of V/V^\perp . The second possibility is that V^\perp is a nonsingular point. Here, one can only induce the bilinear form on V/V^\perp , and this leads to the well-known isomorphism

$$(2.4) \quad \text{Isom}(V_\varphi) \cong \text{Sp}(V/V^\perp).$$

Henceforth, when we say V_φ is an orthogonal space, it is assumed that $\dim V^\perp \leq 1$.

3. SYMMETRIES OF ORTHOGONAL SPACES

We shall not require a detailed description of the involution classes of the symplectic and orthogonal groups; we need just one special type. If $x \in V$ with $\varphi(x) \neq 0$, then

$$(3.1) \quad \sigma_x : v \mapsto v + \frac{(v, x)}{\varphi(x)} x \quad (v \in V)$$

is an involutory isometry V_φ known as a *symmetry*. For $\alpha \in \mathbb{F}_q$, note $\sigma_{\alpha x} = \sigma_x$, so σ_x is uniquely determined by the nonsingular 1-space $\langle x \rangle$. The group $\langle \sigma_x \rangle$ is the subgroup of $\text{Isom}(V_\varphi)$ that induces the identity on $\langle x \rangle^\perp$ and on $V/\langle x \rangle$.

Symmetries have very nice properties that recommend them as generators for string C-groups. First, symmetries σ_x and σ_y commute if, and only if, $(x, y) = 0$. Secondly, the support of a symmetry is simply the nonsingular 1-space that defines it, namely $[V, \langle \sigma_x \rangle] = \langle x \rangle$. Finally, it is well known that $\text{Isom}(V_\varphi)$ is generated by symmetries when V_φ is *nonsingular* [T, Theorem 11.39]. However, we shall need more detailed information about the groups generated by symmetries for our broader notion of orthogonal space. For any subset $X \subseteq V$, put

$$(3.2) \quad \Sigma_X := \langle \sigma_x : x \in X \text{ is nonsingular} \rangle,$$

the group generated by the symmetries determined by nonsingular points in X .

Proposition 3.3. *Let V_φ be an orthogonal space, and W_1, W_2 orthogonal subspaces of dimension $n \geq 3$ such that $U := W_1 \cap W_2$ is an orthogonal subspace of dimension $n - 1$. Then*

$$\Sigma_{W_1} \cap \Sigma_{W_2} = \Sigma_U.$$

Proof. Put $G := \text{Isom}(V_\varphi)$ and, for $i = 1, 2$, put $H_i := \Sigma_{W_i}$. Note, $H_i \leq \text{stab}_G(W_i)$, so $H_1 \cap H_2 \leq \text{stab}_{H_1}(U) \cap \text{stab}_{H_2}(U)$. Evidently $\Sigma_U \leq H_1 \cap H_2$, so we must establish the reverse inclusion. Fix $i \in \{1, 2\}$, and consider the stabilizer $J_i := \text{stab}_{H_i}(U)$.

First, suppose n is even, so that $\text{rad}(W_i) = 0$, $\text{rad}(U) = \langle z \rangle$ for some $z \in U$, and $J_i = \text{stab}_{H_i}(\langle z \rangle)$. If z is nonsingular, $J_i = \langle \sigma_z \rangle \times \text{Isom}(U) = \Sigma_U$, so

$$H_1 \cap H_2 = J_1 \cap J_2 = \langle \sigma_z \rangle \times \text{Isom}(U) = \Sigma_U.$$

Thus, we may assume z is singular. Fix a nondegenerate $(n - 2)$ -space $U_0 \leq U$ not containing z , and choose a singular vector $e_i \in (W_i \cap U_0^\perp) - U$ such that $(z, e_i) = 1$. Then J_i factorizes as $J_i = Q \rtimes \text{stab}_{J_i}(\langle e_i \rangle)$, where $Q = O_2(J_i)$ is the group inducing the identity on $U/\langle z \rangle$. Furthermore, $\text{stab}_{J_i}(\langle e_i \rangle) = \langle h_i \rangle \times K$, where h_i maps $z \mapsto \alpha z$, $e_i \mapsto \alpha^{-1}e_i$ and is the identity on U_0 , and $K \cong \text{Isom}(U_0)$ is the subgroup of J_i inducing the identity on $\langle z, e_i \rangle$. We have omitted the subscripts on Q and K since these groups are common to both J_1 and J_2 , while $h_i \notin J_{3-i}$. Indeed, $J_1 \cap J_2 = Q \rtimes K$, so it suffices to show that $Q \rtimes K \leq \Sigma_U$.

Since U_0 is nondegenerate, $K = \Sigma_{U_0}$. Note, if $x \in U - U_0$ nonsingular, then $\sigma_x \in \Sigma_U$ does not normalize K . As Q acts transitively on its set of complements in $Q \rtimes K$, there exists $1 \neq b \in Q$ such that $K^b = K^{\sigma_x}$, so $b\sigma_x$ normalizes K . Since $N_{Q \rtimes K}(K) = K$, it follows that $b\sigma_x \in K$, so $b \in \langle K, \sigma_x \rangle \leq \Sigma_U$. Finally, Q is isomorphic to the natural orthogonal K -module U_0 , so $Q \rtimes K = \langle K, b \rangle \leq \Sigma_U$.

Next, suppose n is odd. Then W_i has a 1-dimensional radical $\langle z_i \rangle$ that can be singular or nonsingular, and U is nondegenerate. If z_i is nonsingular, then $H_i = \langle \sigma_{z_i} \rangle \times \text{Isom}(W_i)$ and $J_i = \langle \sigma_{z_i} \rangle \times \Sigma_U$, where $\Sigma_U \cong \text{Isom}(U)$. If z_i is singular then, as in the singular case above, $H_i = O_2(H_i) \rtimes \text{stab}_{H_i}(U)$, so that $J_i = \text{stab}_{H_i}(U) = \Sigma_U \cong \text{Isom}(U)$. There are now three cases to consider: z_1 and z_2 are both nonsingular, both singular, or one is singular and one is nonsingular. Observe, however, if z_i is nonsingular then $\sigma_{z_i} \notin J_{3-i}$; thus, it is clear from the structure of J_i that $J_1 \cap J_2 = \Sigma_U$ in all three cases. \square

We will also make use of the following elementary result on generation by symmetries in orthogonal 3-spaces.

Lemma 3.4. *If $q \geq 4$, V_φ is a 3-dimensional orthogonal space over \mathbb{F}_q , and L is an asingular line of V , then Σ_L is maximal in Σ_V .*

Proof. If V^\perp is nonsingular, then $\Sigma_V = \text{Isom}(V_\varphi) \cong \text{Sp}(V/V^\perp) \cong \text{SL}(2, \mathbb{F}_q)$ and $\Sigma_L = \text{O}^-(L) \cong D_{2(q+1)}$. It is well known that all $D_{2(q+1)}$ subgroups of $\text{SL}(2, \mathbb{F}_q)$ are maximal when $q \geq 4$. If V^\perp is singular, we have $\text{Isom}(V_\varphi) \cong Q \rtimes (\mathbb{F}_q^\times \times \text{O}^-(L))$, where $Q = \text{O}_2(\text{Isom}(V_\varphi))$. Thus, as in the “ n even, z singular” case in the proof of Proposition 3.3, we have $\Sigma_V \cong Q \rtimes \text{O}^-(L) \cong Q \rtimes \Sigma_L$. In particular, Σ_L is its own normalizer in Σ_V and the regularity of Q on its set of complements in Σ_V ensures that $\langle g, \Sigma_L \rangle = \langle u, \Sigma_L \rangle$ for any $g \in \Sigma_V \setminus \Sigma_L$. Again, Q is the natural module for $\Sigma_L \cong \text{O}^-(L)$ under the conjugation action, so $\langle u, \Sigma_L \rangle = Q \rtimes \Sigma_L = \Sigma_V$. \square

Remark 3.5. If $q = 2$, V^\perp is nonsingular, and L is asingular, then $\Sigma_L \cong D_6 \cong \text{SL}(2, \mathbb{F}_2) \cong \Sigma_V$; the condition $q \geq 4$ is needed. For $q \geq 8$, the result also holds for a hyperbolic line L by the same argument; for $q = 4$, however, there exist elements $u \in Q$ such that $\langle u, \Sigma_L \rangle$ is a proper intermediate subgroup of Σ_L and Σ_V .

4. STRING GROUPS GENERATED BY SYMMETRIES

Let V_φ be an orthogonal space of dimension d over the field \mathbb{F}_q , where $q = 2^k$. Suppose that $(G; \{\sigma_{v_1}, \dots, \sigma_{v_d}\})$ is a string subgroup of $\text{O}(V_\varphi)$ generated by d symmetries. Considering commutator relations among symmetries, we see that $(v_i, v_j) = 0$ if, and only if, $|i - j| \neq 1$. As the v_i are nonsingular vectors and our field has characteristic 2, we can normalize the v_i so that $\varphi(v_i) = 1$. Thus, relative to the basis v_1, \dots, v_d , the quadratic form φ has matrix

$$(4.1) \quad \Phi(\alpha_1, \dots, \alpha_{d-1}) = \begin{bmatrix} 1 & \alpha_1 & & & & \\ & 1 & \alpha_2 & & & \\ & & \ddots & \ddots & & \\ & & & 1 & \alpha_{d-1} & \\ & & & & 1 & \end{bmatrix},$$

for some $\alpha_1, \dots, \alpha_{d-1} \in \mathbb{F}_q$, where all other entries are 0. The bilinear form $(,)$ associated to φ is represented by $B = \Phi + \Phi^{\text{tr}}$. If in addition G has no direct product decomposition, then the scalars α_i belong to \mathbb{F}_q^\times .

Next, consider the matrices representing the symmetries σ_{v_i} for $i = 1, \dots, d$ relative to the same basis. First, σ_{v_1} maps $v_2 \mapsto v_2 + \alpha_1 v_1$ and fixes the remaining basis vectors. It therefore has matrix $\begin{bmatrix} 1 & \\ \alpha_1 & 1 \end{bmatrix} \oplus I_{d-2}$. Similarly, σ_d has matrix $I_{d-2} \oplus \begin{bmatrix} 1 & \alpha_{d-1} \\ & 1 \end{bmatrix}$. For $2 \leq i \leq d-1$, σ_i is represented by a matrix whose restriction to $\langle v_{i-1}, v_i, v_{i+1} \rangle$ has the form

$$\begin{bmatrix} 1 & \alpha_{i-1} & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \alpha_i & 1 \end{bmatrix},$$

inducing the identity on the span of the remaining basis vectors.

Conversely, any choice of scalars $\alpha_1, \dots, \alpha_{d-1} \in \mathbb{F}_q^\times$ determines both a quadratic form φ on $V = \mathbb{F}_q^d$ represented by the matrix $\Phi(\alpha_1, \dots, \alpha_{d-1})$, and a string group $(G; \{\rho_1, \dots, \rho_d\})$, where ρ_i is the symmetry determined by the i^{th} (nonsingular) standard basis vector of V . The properties of both V_φ and the string group $(G; \{\rho_1, \dots, \rho_d\})$ are determined by the scalars α_i .

In order to prove Theorem 1.4 we shall need to demonstrate that the scalars $\alpha_1, \dots, \alpha_{d-1}$ may be selected so that $(G; \{\rho_1, \dots, \rho_d\})$ also satisfies the intersection property needed to make it a string C-group. We take up this issue in the next section. To prove Corollary 1.5, we must also understand the radical of an orthogonal space V_φ having matrix $\Phi(\alpha_1, \dots, \alpha_{d-1})$ and, for nondegenerate V_φ in even dimension, how the choice of scalars affects the Witt index of the space. The following characterization of the radical of V_φ solves the first of these problems.

Lemma 4.2. *If $V = \mathbb{F}_q^d$, $\alpha_1, \dots, \alpha_{d-1}$ are any elements of $\mathbb{F}_{2^k}^\times$, and φ is the quadratic form represented by $\Phi = \Phi(\alpha_1, \dots, \alpha_{d-1})$, then V_φ is an orthogonal space. In particular, if d is even then V_φ is nondegenerate, and if $d = 2m + 1$ is odd then V_φ has 1-dimensional radical spanned by*

$$(4.3) \quad z = (1, 0, \beta_1, 0, \dots, 0, \beta_m), \quad \text{where } \beta_s = \prod_{i=1}^s \left(\frac{\alpha_{2i-1}}{\alpha_{2i}} \right) \text{ for } s = 1, \dots, m.$$

Furthermore, the radical is singular precisely when $1 + \beta_1^2 + \dots + \beta_m^2 = 0$.

Proof. This is a direct calculation. Recall that the matrix $B = \Phi + \Phi^{\text{tr}}$ represents the symmetric form determined by Φ , and $\langle z \rangle$ is its nullspace. Thus, whether V^\perp is singular or nonsingular is determined by whether $\varphi(z) = z\Phi z^{\text{tr}}$ is zero or nonzero, respectively; this gives rise to the condition in the lemma. \square

Suppose that d is even, and let $\Phi = \Phi(\alpha_1, \dots, \alpha_{d-1})$ for some $\alpha_i \in \mathbb{F}_q^\times$. Then, by Lemma 4.2, the orthogonal space V_φ is nondegenerate. We now determine the Witt index of V_φ . Let $e_1, f_1, \dots, e_m, f_m$ be any hyperbolic basis of the symplectic space V associated with V_φ . Let N denote the additive subgroup of \mathbb{F}_q^+ consisting of all elements $\alpha^2 + \alpha$, where $\alpha \in \mathbb{F}_q$. The *Arf invariant* of V_φ is

$$(4.4) \quad \text{Arf}(V_\varphi) = \sum_{i=1}^m \varphi(e_i)\varphi(f_i) \pmod{N},$$

an element of $\mathbb{F}_q^+/N \cong \mathbb{Z}_2$. The Witt index of V_φ is m or $m-1$ according as $\text{Arf}(V_\varphi)$ is 0 or 1 mod N , respectively [A]. Thus, we next compute a hyperbolic basis for $(,)$ by applying the Gram-Schmidt process to $B = \Phi + \Phi^{\text{tr}}$.

For successive $i = 0, \dots, 1 + (d-4)/2$, add $\alpha_{2i+2}/\alpha_{2i+1}$ times row-column $2i+1$ to row-column $2i+3$. This process reduces B to the matrix

$$\begin{bmatrix} 0 & \alpha_1 \\ \alpha_1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & \alpha_3 \\ \alpha_3 & 0 \end{bmatrix} \oplus \dots \oplus \begin{bmatrix} 0 & \alpha_{d-1} \\ \alpha_{d-1} & 0 \end{bmatrix},$$

which is almost a hyperbolic basis. Indeed, scaling alternate vectors in the resulting basis, we obtain the desired hyperbolic basis $e_1, f_1, \dots, e_m, f_m$, where $e_1 = v_1$,

$$(4.5) \quad e_i = v_{2i-1} + \sum_{j=1}^{i-1} \left(\prod_{\ell=1}^j \frac{\alpha_{2i-2\ell}}{\alpha_{2i-2\ell-1}} \right) v_{2i-2j-1} \quad \text{for } 2 \leq i \leq m,$$

and $f_i = v_{2i}/\alpha_{2i-1}$ for $1 \leq i \leq m$. Although this is a rather unwieldy formula, we can simplify it greatly through judicious choices of scalars. We shall return to this when we prove Corollary 1.5 in the following section.

5. PROOFS OF THE MAIN RESULTS

We saw in the previous section that any selection of scalars $\alpha_1, \dots, \alpha_{d-1}$ from \mathbb{F}_q^\times defines a quadratic form φ represented on the row space $V = \mathbb{F}_q^d$ by $\Phi(\alpha_1, \dots, \alpha_{d-1})$, turning V_φ into an orthogonal space. If v_1, \dots, v_d denotes the standard basis of V , we also saw that $G = \langle \sigma_{v_1}, \dots, \sigma_{v_d} \rangle$ is a string group. We now show that the scalars α_i may be chosen so that $G = \mathcal{O}(V_\varphi)$ and $(G; \{\sigma_{v_1}, \dots, \sigma_{v_d}\})$ is a string C-group, thereby proving Theorem 1.4.

Fix $1 \leq i \leq d-1$, and consider the dihedral group $\langle \sigma_{v_i}, \sigma_{v_{i+1}} \rangle$, which acts on its 2-dimensional support $\langle v_i, v_{i+1} \rangle$. We select the scalar α_i so that two conditions are satisfied: first that $\langle v_i, v_{i+1} \rangle$ is asingular, and secondly that $\langle \sigma_{v_i}, \sigma_{v_{i+1}} \rangle$ induces the full group $\mathcal{O}^-(\langle v_i, v_{i+1} \rangle)$, of order $2(q+1)$, on this 2-space. The restriction of φ to $\langle v_i, v_{i+1} \rangle$ is represented by the matrix $\begin{bmatrix} 1 & \alpha_i \\ 0 & 1 \end{bmatrix}$; the Arf invariant of this form is $\alpha_i^{-1} \pmod{N}$. Thus, the first condition is satisfied so long as all α_i are chosen from the set

$$(5.1) \quad A_0 = \{\beta \in \mathbb{F}_q^* : \beta^{-1} \notin N\},$$

of size $\frac{q}{2}$. Next, the restriction of the product $h_i := \sigma_{v_i} \sigma_{v_{i+1}}$ has matrix

$$(5.2) \quad H_{\alpha_i} := \begin{bmatrix} 1 & \alpha_i \\ \alpha_i & 1 + \alpha_i^2 \end{bmatrix} \in \mathrm{SL}(2, \mathbb{F}_q).$$

For α_i selected from A_0 , the order of this matrix divides $q+1$. Thus, we wish to select our scalars from the set $A = \{\beta \in A_0 : H_\beta \text{ has order } q+1\}$. The proportion of elements of A_0 that belong to A is determined by the Euler totient of $q+1$.

The following somewhat technical result is crucial to our proof of Theorem 1.4.

Lemma 5.3. *Let $q = 2^k \geq 4$ and V a vector space of dimension d over \mathbb{F}_q . Let $\alpha_1, \dots, \alpha_{d-1} \in A$, and let φ be the quadratic form represented in a basis v_1, \dots, v_d of V by the matrix $\Phi(\alpha_1, \dots, \alpha_{d-1})$ in equation (4.1). For each $1 \leq b \leq d-1$, and each $1 \leq i \leq d-b+1$, if*

$$X_{b,i} = \{v_i, \dots, v_{i+(b-1)}\}, \text{ then } \Sigma_{X_{b,i}} = \Sigma_{\langle X_{b,i} \rangle}.$$

In words, the group generated by the symmetries determined by consecutive basis vectors always coincides with the (potentially larger) group associated with the linear span of these vectors.

Proof. We proceed by induction on b , with the case $b=1$ being obvious.

When $b=2$, we are given nonsingular v_i, v_{i+1} such that $\langle v_i, v_{i+1} \rangle$ is asingular and $h := \sigma_{v_i} \sigma_{v_{i+1}}$ has order $q+1$, and we wish to show that $\Sigma_{\{v_i, v_{i+1}\}} = \Sigma_{\langle v_i, v_{i+1} \rangle}$. Note, $\langle h \rangle$ acts regularly on the set of points of $\langle v_i, v_{i+1} \rangle$. Thus, for each nonzero vector x in $\langle v_i, v_{i+1} \rangle$, there exists $0 \leq j < q+1$ such that $\langle x \rangle = \langle v_i \rangle h^j$. Hence, $\sigma_x = \sigma_{v_i}^{h^j} \in \Sigma_{\{v_i, v_{i+1}\}}$, as required.

The case $b=3$ follows from $b=2$ together with Lemma 3.4. For, if $L = \langle v_i, v_{i+1} \rangle$, then $\Sigma_L = \Sigma_{X_{2,i}} \leq \Sigma_{X_{3,i}}$ by the $b=2$ case. By Lemma 3.4, Σ_L is maximal in $\Sigma_{\langle X_{3,i} \rangle}$. As $\sigma_{v_{i+2}} \notin \Sigma_L$, so $\Sigma_{\langle X_{3,i} \rangle} = \Sigma_{L \cup \{v_{i+2}\}} \leq \Sigma_{X_{3,i}}$. (Note, this argument applies to the \mathbb{F}_2 case as well, since φ has a singular radical.)

We can use Lemma 3.4 to treat the inductive $b > 2$ case uniformly as follows. Consider $y \in \langle X_{b,i} \rangle$ nonsingular; we must show that $\sigma_y \in \Sigma_{X_{b,i}}$. Let $W = \langle X_{b-1, i+1} \rangle$. By induction, $\Sigma_W = \Sigma_{X_{b-1, i+1}} \leq \Sigma_{X_{b,i}}$, so we may assume that

$\langle y \rangle$ lies off W . Similarly, if $L = \langle v_i, v_{i+1} \rangle$, then $\Sigma_L = \Sigma_{X_{2,i}} \leq \Sigma_{X_{b,i}}$ by the $b = 2$ case, so we may further assume that $\langle y \rangle$ lies off L . Thus, the plane $U = \langle v_i, v_{i+1}, y \rangle$ meets W in a line, M , containing $\langle v_{i+1} \rangle$. As M is not totally singular and contains $q \geq 4$ points not equal to $\langle v_{i+1} \rangle$, at least one of those, x , is nonsingular. Evidently, $\sigma_x \in \Sigma_W \in \Sigma_{X_{b,i}}$. Now, by the argument in the foregoing paragraph, it follows that $\Sigma_U = \Sigma_{L \cup \{x\}} \leq \Sigma_{X_{b,i}}$. In particular $\sigma_y \in \Sigma_{X_{b,i}}$, as claimed. \square

Remark 5.4. We might equally have chosen the α_i so that the matrices in (5.2) have order $q - 1$ or any combination of $q - 1$ and $q + 1$. The crucial property is that $\langle \sigma_{v_i}, \sigma_{v_{i+1}} \rangle$ induces the full group of isometries on its support $\langle v_i, v_{i+1} \rangle$. There is one exception: when $q = 4$, strings $\sigma_{v_i}, \dots, \sigma_{v_{i+t}}$ such that $\sigma_{v_j} \sigma_{v_{j+1}}$ has order $q - 1 = 3$ generate groups S_{t+1} rather than $O(\langle v_i, \dots, v_{i+t} \rangle)$; cf. Remark 3.5. We insisted on using elements of order $q + 1$ to avoid dealing with this exception.

We shall need to check that certain sequences of involutions satisfy the intersection property (1.1). This would be quite tedious if one had to check all possible intersections, but the following result makes the task tractable.

Proposition 5.5. [MS1, Proposition 2E16] *Let G be a group generated by involutions ρ_1, \dots, ρ_n . If both $\langle \rho_1, \dots, \rho_{n-1} \rangle$ and $\langle \rho_2, \dots, \rho_n \rangle$ are string C-groups on their defining generators, and*

$$\langle \rho_1, \dots, \rho_{n-1} \rangle \cap \langle \rho_2, \dots, \rho_n \rangle = \langle \rho_2, \dots, \rho_{n-1} \rangle,$$

then ρ_1, \dots, ρ_n satisfies the intersection property. In particular, $(G; \{\rho_1, \dots, \rho_n\})$ is a string C-group.

Proof of Theorem 1.4. Let $q = 2^k$ for $k \geq 2$, and $V = \mathbb{F}_q^d$ for $d \geq 2$. To begin with, choose any scalars $\alpha_1, \dots, \alpha_{d-1} \in A$. Let φ be the quadratic form on V represented relative to a basis v_1, \dots, v_d by the matrix $\Phi(\alpha_1, \dots, \alpha_{d-1})$ in equation (4.1). We show that $(\Sigma_V; \{\sigma_{v_1}, \dots, \sigma_{v_d}\})$ is a string C-group. The sequence $\sigma_{v_1}, \dots, \sigma_{v_d}$ satisfies the necessary string condition by construction and they generate Σ_V by Lemma 5.3. It remains only to establish the intersection property.

We proceed by induction on d , following the notation of Lemma 5.3. Recall, $\langle \sigma_{v_1}, \sigma_{v_2} \rangle$ is the dihedral group $D_{2(q+1)} \cong O^-(2, \mathbb{F}_q)$ so the result is clear for the base case $d = 2$. Let $d > 2$ and suppose that $\Sigma_{X_{b,i}}$ (with its defining generators) satisfies the intersection property whenever $b < d$. In particular, both $\Sigma_{X_{d-1,1}}$ and $\Sigma_{X_{d-1,2}}$ satisfy the intersection property. Note, $\langle X_{d-1,1} \rangle \cap \langle X_{d-1,2} \rangle = \langle X_{d-2,2} \rangle$, so it follows from Lemma 5.3 and Proposition 3.3 that

$$\Sigma_{X_{d-1,1}} \cap \Sigma_{X_{d-1,2}} = \Sigma_{\langle X_{d-1,1} \rangle} \cap \Sigma_{\langle X_{d-1,2} \rangle} = \Sigma_{\langle X_{d-2,2} \rangle} = \Sigma_{X_{d-2,2}}.$$

The intersection property now follows from Proposition 5.5.

When $q = 2$, $A = \{1\}$ so there is only one choice for each of the scalars α_i . Here, as in Remark 5.4, each $\sigma_{v_i} \sigma_{v_{i+1}}$ has order 3, so $\langle \sigma_{v_1}, \dots, \sigma_{v_d} \rangle \cong S_d$. Thus, it is not possible to generate $O(d, \mathbb{F}_2)$ as a string C-group using symmetries. \square

Proof of Corollary 1.5. First, let $d = 2m$. We must show that, given either possible isometry type $\epsilon \in \{+, -\}$ of a quadratic space $V = \mathbb{F}_q^d$, the scalars $\alpha_1, \dots, \alpha_{d-1} \in A$ may be chosen to endow V with a form $\Phi(\alpha_1, \dots, \alpha_{d-1})$ of type ϵ . To this end, we employ the Arf invariant described in the preceding section.

For $\lambda, \mu \in A$, put $\alpha_1 = \alpha_{d-1} = \lambda$ and $\alpha_i = \mu$ for $1 < i < d-1$. Using equations (4.4) and (4.5), we compute

$$(5.6) \quad \text{Arf}(V_\varphi) = \frac{m(m-1)}{2} \cdot \mu^{-1} + (m-1) \cdot \lambda^{-1} + \frac{\mu}{\lambda^2} \pmod{N}.$$

If $\mu = \lambda$, then (5.6) reduces to $\text{Arf}(V_\varphi) = \frac{m(m+1)}{2} \cdot \lambda^{-1} \pmod{N}$. As $\lambda \in A$, so $\lambda^{-1} \notin N$; hence, in this case $\text{Arf}(V_\varphi) = 0$ if, and only if, m is congruent to 0 or 3 mod 4. If, on the other hand, we choose μ so that $\frac{\mu}{\lambda^2} \in N$, then $\text{Arf}(V_\varphi) = 0$ if, and only if, m is congruent to 1 or 2 mod 4. Hence, whatever the value of m , we can construct a suitable quadratic form on $\mathbb{F}_{2^k}^{2m}$ of either isometry type. In particular, $O^\pm(2m, \mathbb{F}_{2^k})$ is always a string C-group of rank $2m$, as claimed.

Next, take $d = 2m + 1$ to be odd. Referring to Lemma 4.2, the 1-dimensional radical V^\perp is nonsingular precisely when $1 + \beta_1^2 + \dots + \beta_m^2 \neq 0$. Observe that the value of β_m^2 is changed if one fixes all α_i for $1 \leq i < d-1$ and makes a different choice for $\alpha_{2m} = \alpha_{d-1}$. Hence, we can always arrange for V^\perp to be nonsingular. Having done so, we have $\text{Sp}(2m, \mathbb{F}_q) \cong \text{Sp}(V/V^\perp) \cong \text{Isom}(V)$, and the result follows. \square

6. LOW-DIMENSIONAL CLASSICAL GROUPS ACTING ON POLYTOPES

There has been a significant effort in recent years to determine the pairs (G, r) such that the finite simple group G is a string C-group of rank r . The question has been settled completely for $r = 3$ by the combined results of Nuzhin for simple groups of Lie type [N1, N2] and Mazurov for sporadic simple groups [M].

For higher ranks, the question has also been settled for alternating groups in three separate papers involving the third author [FLM2, FLM3, CFLM]. For other types of simple groups the study has generally progressed in a more piecemeal fashion by considering individual infinite families such as $\text{PSL}(2, \mathbb{F}_q)$, $\text{PSL}(3, \mathbb{F}_q)$, and $\text{PSL}(4, \mathbb{F}_q)$ (see [LS, BV, BL], respectively). The results in this paper, as well as contributing generally to the ongoing effort to understand abstract regular polytopes of higher rank, help to settle the string C-group question for low-dimensional simple classical groups defined over finite fields of characteristic 2.

Theorem 6.1. *Let \mathbb{F}_q be the finite field with $q = 2^k$ elements, V an \mathbb{F}_q -vector space of dimension at most 4, and G a quasisimple classical group whose natural module is V . Then the associated simple group $G/Z(G)$ is a string C-group of rank $r \geq 4$ if, and only if, $r \in \{4, 5\}$ and $G = \text{Sp}(4, \mathbb{F}_q)$.*

Proof. We first show that no quasisimple classical group G other than $\text{Sp}(4, \mathbb{F}_q)$ is such that $G/Z(G)$ is a string C-group of rank $r \geq 4$. First, Leemans and Schulte [LS] showed that $\text{PSL}(2, \mathbb{F}_q)$ is a string C-group of rank $r \geq 4$ if, and only if, $r = 4$ and $q \in \{11, 19\}$. Secondly, Brooksbank and Vicinsky [BV] showed that if G is an irreducible string subgroup of $\text{GL}(3, \mathbb{F}_q)$, then G preserves a symmetric bilinear form, so $G/Z(G) \cong \text{PSL}(2, \mathbb{F}_q)$. Finally, Brooksbank and Leemans [BL, Corollary 4.5] showed that $\text{PSL}(4, \mathbb{F}_{2^k})$ is not a string C-group of any rank. The latter was done by showing that the isomorphic group $\Omega^+(6, \mathbb{F}_{2^k})$ is not a string C-group of any rank, and the exact same proof applies to $\Omega^-(6, \mathbb{F}_{2^k}) \cong \text{PSU}(4, \mathbb{F}_{2^k})$.

Next, suppose that $G = \text{Sp}(4, \mathbb{F}_{2^k})$; note $Z(G) = 1$, so G is simple. We may assume that $k \geq 2$, since $\text{Sp}(4, \mathbb{F}_2) \cong \text{Sym}(6)$ is not quasisimple. (Moreover, the derived group $\text{Sp}(4, \mathbb{F}_2)' \cong \text{Alt}(6)$ is not a string C-group of rank $r \geq 4$.) By Corollary 1.5, G is a string C-group of rank 5. However, G is not a string C-group

of rank $r \geq 6$ since none of its maximal subgroups contain a string C-group of rank $r - 1 \geq 5$; see [BHRD, Table 8.14]. It therefore remains to show that G is also a string C-group of rank 4, which we do by constructing it as such.

As usual, let $(,)$ be a nondegenerate symmetric bilinear form on V , and φ a nondegenerate quadratic form of Witt index 1 whose associated symmetric form is $(,)$. Let $L = \langle e, b \rangle$ be a 2-dimensional subspace of the orthogonal space V_φ with $\varphi(e) = (e, b) = 0$ and $\varphi(b) \neq 0$. Fix $0 \neq \alpha \in \mathbb{F}_q$ and let ρ be the *pseudo-transvection*

$$\rho: v \mapsto v + \alpha[(v, b)e + (v, e)b] + \alpha^2\varphi(b)(v, e)e.$$

Consider nonsingular vectors $u, w \notin \langle e \rangle^\perp$ with $(u, w) = 0$; we will refine the selection of u, w subject to these fundamental conditions. Note, $U = \langle u, L \rangle$ is a 3-space with nonsingular radical $\langle x \rangle = \langle u \rangle^\perp \cap L$. Similarly, $W = \langle w, L \rangle$ has nonsingular radical $\langle y \rangle = \langle w \rangle^\perp \cap L$.

Let us consider U . Note that ρ induces the identity on L , but moves $\langle u \rangle$. Hence, $\langle \sigma_u, \rho \rangle$ fixes both the radical $\langle x \rangle$ of U and the line $\langle u, u\rho \rangle$. By a suitable choice of u , one can arrange for $\langle u, u\rho \rangle$ to be asingular, and for the group induced on $\langle u, u\rho \rangle$ by $\langle \sigma_u, \rho \rangle$ to be the full group $O^-(\langle u, u\rho \rangle) \cong D_{2(q+1)}$ of isometries. Indeed, for such u we have $\langle \sigma_u, \rho \rangle \cong O^-(\langle u, u\rho \rangle) \times \langle \sigma_x \rangle \cong D_{4(q+1)}$.

In the same way, we can select w so that $\langle \sigma_w, \rho \rangle \cong O^-(\langle w, w\rho \rangle) \times \langle \sigma_y \rangle \cong D_{4(q+1)}$. Note that the condition $(u, w) = 0$ ensures that $[\sigma_u, \sigma_w] = 1$, so $\langle \sigma_u, \rho, \sigma_w \rangle$ is a string group. Furthermore, we have $\rho \in O(V_\varphi)' = \Omega(V_\varphi)$, while $\sigma_u, \sigma_w \in O(V_\varphi) \setminus \Omega(V_\varphi)$. Hence, $(\sigma_u\rho)^2$ and $(\sigma_w\rho)^2$ are non-commuting elements of order $q+1$ in $\Omega^-(4, \mathbb{F}_q) \cong \text{SL}(2, \mathbb{F}_{q^2})$, and so generate this group. It follows that $\langle \sigma_u, \rho, \sigma_w \rangle \cong O^-(V_\varphi)$.

Consider the group T of transvections inducing the identity on $\langle x \rangle^\perp = U$ and on $V/\langle x \rangle$. Note that $\sigma_x \in T \cong \mathbb{F}_q^+$ and is the only element of this group belonging to $O(V_\varphi)$. As $q \geq 4$, there exists $\tau \in T \setminus \{1, \sigma_x\}$. The choice of τ ensures that $\langle \sigma_u\rho, \sigma_w, \tau \rangle$ is a string group. Further, since $O^-(V_\varphi)$ is maximal in $\text{Sp}(V)$ and $\tau \notin O^-(V_\varphi)$, we have $\text{Sp}(V) = \langle \sigma_u, \rho, \sigma_w, \tau \rangle$.

Next, observe that $\langle \rho, \sigma_w, \tau \rangle$ fixes the plane W , and hence $W^\perp = \langle y \rangle$. Indeed, $\langle \rho, \sigma_w, \tau \rangle$ preserves a quadratic form φ' on W distinct from the restriction of φ , and the structure of the group is determined by whether or not $\langle y \rangle$ is nonsingular with respect to this form, as described in the proof of Lemma 3.4. For us, what matters in each case is the intersection of that group with $\langle \sigma_u, \rho, \sigma_w \rangle$. To that end, we consider the stabilizer in $\langle \rho, \sigma_w, \tau \rangle$ of the line $\langle w, w\rho \rangle$. If $\varphi'(y) \neq 0$, then the stabilizer is $\langle \rho, \sigma_w \rangle \times \langle \sigma'_y \rangle$, where σ'_y is the symmetry determined by y relative to φ' . This is an element of the group of transvections inducing the identity on $\langle y \rangle^\perp$ and $V/\langle y \rangle$, but is distinct from $\sigma_y \in \langle \rho, \sigma_w \rangle$, so $\sigma'_y \notin \langle \sigma_u, \rho, \sigma_w \rangle$. If $\varphi'(y) = 0$, on the other hand, then the stabilizer of $\langle w, w\rho \rangle$ is simply $\langle \rho, \sigma_w \rangle$. Thus, in each case

$$\langle \sigma_u, \rho, \sigma_w \rangle \cap \langle \rho, \sigma_w, \tau \rangle = \langle \rho, \sigma_w \rangle,$$

so $(\text{Sp}(V); \{\sigma_u, \rho, \sigma_w, \tau\})$ satisfies the intersection property by Proposition 5.5 and is therefore a string C-group, as required. \square

For completeness, we describe suitable involutions $\sigma_u, \rho, \sigma_w, \tau$ explicitly as matrices relative to the ordered basis u, e, b, w . Let $\alpha \in \mathbb{F}_q^\times$ be such that $\begin{bmatrix} 0 & 1 \\ 1 & \alpha^2 \end{bmatrix}$ has order $q+1$. Now, put

$$\sigma_u := \begin{bmatrix} 1 & 0 & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \rho := \begin{bmatrix} 1 & \alpha & \alpha & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1+\alpha & \alpha & 1 \end{bmatrix}, \quad \sigma_w := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \tau := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \alpha & 1 \end{bmatrix}.$$

7. CONCLUDING REMARKS

To the reader interested in computer experimentation with abstract regular polytopes, we mention that MAGMA code is available upon request from the authors to construct the generators exhibiting $O^\pm(d, \mathbb{F}_{2^k})$ as string C-groups of rank d , and $Sp(d, \mathbb{F}_{2^k})$ as a string C-group of rank $d + 1$.

As noted at the start of the preceding section, a problem of general interest is to determine the pairs (G, r) such that the finite simple group G is a string C-group of rank r . The main results in this paper provide partial answers for the simple groups $Sp(d, \mathbb{F}_{2^k})$ and the almost simple groups $O^\pm(d, \mathbb{F}_{2^k})$, in particular showing that we can build abstract regular polytopes of arbitrarily high rank whose group of automorphisms is an orthogonal or symplectic group. In proving Theorem 6.1, we further showed that as well as having polytopes of rank 5 the simple groups $Sp(4, \mathbb{F}_{2^k})$ have polytopes of rank 4.

Many open questions remain. First, are the orthogonal groups $O(d, \mathbb{F}_2)$ string C-groups? Theorem 1.4 shows they cannot be constructed as such just using symmetries, but the proof of Theorem 6.1 shows how other types of involutions can be exploited. (Computer experiments provide evidence that $O(d, \mathbb{F}_2)$ is a string C-group of rank $d - 1$.) Secondly, are the simple groups $\Omega^\pm(2m, \mathbb{F}_{2^k})$ string C-groups for $m \geq 3$? In the proof of Theorem 6.1 we remarked that $\Omega^\pm(6, \mathbb{F}_{2^k})$ are *not* string C-groups of any rank. The proof of this fact in [BL] relied on the very specific nature of the involutions in these groups and the consequent geometric constraints imposed upon strings of involutions. In higher dimensions there is a greater diversity of involution classes in $\Omega^\pm(2m, \mathbb{F}_{2^k})$ and it is not known whether they can be generated as string C-groups.

Perhaps the most interesting question in this line, though, is whether or not the most fundamental family of finite simple classical groups—the projective special linear groups—have polytopes of arbitrarily high rank. The first and third authors showed in [BL] that $PSL(4, \mathbb{F}_q)$ have polytopes of rank 4 when q is odd, but no other constructions are known.

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