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Centers of n-degree Poncelet Circles

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by

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Abstract

Given a convex polygon inscribed in a circle that circumscribes a second circle, connect each pair of vertices with a line segment. A theorem of Poncelet tells us that every point on the circle is a vertex of one such polygon. In certain cases, these polygons give rise to a family of circles called a **package of Poncelet circles.** Specifically, we are interested in the centers of these circles.

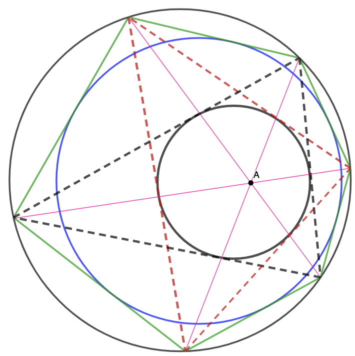

Figure 1: Package of Poncelet circles, $m = 6$

In particular, we wish to answer the question: *if we know where one of the centers of the circles in the package is, can we know where all of the centers are as well?* These families of circles are also connected to a set called the **numerical range** that is related to an operator (i.e., a linear function that maps vector spaces to vector spaces). The numerical range of an operator is a convex set in the complex plane $\mathbb{C}$ that contains the eigenvalues of the operator as well as other information about the operator.

An open question in operator theory ([PT02], pp. 7) asks: *when the numerical range of an operator is circular?* In 2016, Gau, Wang, and Wu [GWW16] looked at a special class of operators called **partial isometries,** which are operators that preserve distance on a specific subset of the domain. They showed that if the numerical range of a partial isometry on a space of dimension $n \leq 4$ is circular, then it must be

centered at the origin. Their proof goes through each dimension $n = 2, 3, 4$ and for each n, they prove this result for all possible dimensions of the kernel of the operator. Our work simplifies and clarifies each proof.

In 2021, Spitkovsky and Wegert [WS21] proved the same statement for all n for a certain class of partial isometries whose dimension of the kernel is 1. Their proof involves elliptic integrals and requires a high level of mathematical understanding to follow. Our work re-proves Spitkovsky and Wegert's theorem using information gleaned from the package of Poncelet circles. We show that if the one of the circles in the package is centered at the origin, then all of the circles must be centered at the origin. We do this in two ways. First, we have a straight forward projective geometric proof that follows the work done by Mirman closely, but it hides what is happening geometrically. So we also found a classical geometric proof of this result.

1 Introduction

1.1 Motivation

Circles inscribed in triangles and squares have been studied for centuries. For instance, we have the Euler-Chapple formula theorem that was published in 1765 for a circle inscribed in a triangle inscribed in another circle that gives a relation between the centers and radii.

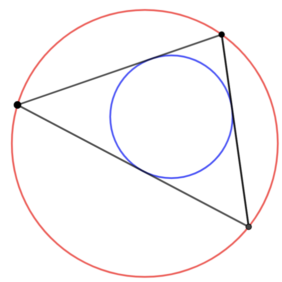

Figure 2: Euler-Chapple theorem

Theorem 1 *([AC07] pp. 85)*

Let C_1 be a circle of radius R and suppose there is a triangle that has its three vertices on C_1. Let $r < R$ be the radius of a circle inside of the triangle and tangent to each side of the triangle. Let d be the distance between the centers of the circles. Then

$$d^2 = R(R - 2r).$$

Circles inscribed in polygons also appear in various real world examples, one of which is in architecture (see Figure 3). Further, we see them in philosophical ideas. Circles have been associated with that of divinity due to the "infinite" nature of a circle. And a square, with its four sides, is finite and "earthly." So the combination of circles and squares has spiritual meaning as the connection between celestial beings and those

 

Figure 3: Left: Calakmul building in México City. México. Designed by Agustín Hernandez; Right: Phillips Exeter Academy Library in Exeter NH, USA. Designed by Louis I Khan.

of us on earth. Lastly, we find circles in squares in numerous pieces of art work (see Figure 4).

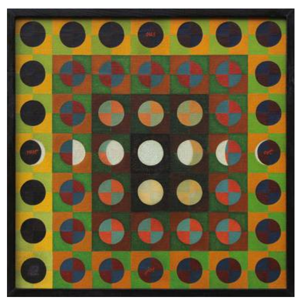

Figure 4: Inspre-Dinspre (Towards-From there) painting by Serge Vasilendiuc

Understanding more about polygons inscribed in circles is a natural question to ask and it is not surprising that mathematicians studied them.

One such mathematician was Jean-Victor Poncelet who was born in France in 1788. Like many famous mathematicians, he lived an interesting life. He joined the Engineering Corps in France and ultimately joined Napoleon's army. He became a prisoner of war from 1813-1814 and during his imprisonment, he looked at circles inscribed in polygons and found his most famous theorem now called *Poncelet's closure theorem*, which we state in Section 1.2. After his release, he published *Traité sur les propriétés projectives des figures* in 1822 where he shared this theorem with the world. Following this publication, several other mathematicians studied, generalized, and re-proved the result using different approaches. A few of the mathematicians who extended Poncelet's work are: Carl Gustav Jacob Jacobi, Arthur Cayley, Gaston Darboux, and George H. Halphen. Six years after Poncelet's publication, Jacobi published a result connecting this problem to elliptic functions. Then, in 1853, Cayley found explicit conditions that allow the existence of an n-gon inscribed in one conic and circumscribing another. Towards the late nineteenth century, Darboux took a geometric approach to Poncelet's theorems and published his results in 1870. However, he spent the next half century refining and perfecting his results, which he published in 1917. Lastly, unlike Darboux, Halphen went back to elliptic functions to prove Poncelet's Theorem, where his main contribution was the addition of explicit elliptic curves in the "elliptic representation" of points of the plane.

Note that we have only chosen a few mathematicians to highlight and this is not a comprehensive list of the mathematicians who have contributed to the discussion around Poncelet's theorem. For those interested in an in-depth historical account of Poncelet's theorem, it can be found in [DC16a], [DC16b].

Our work continues this discussion on Poncelet's Theorem through a connection of Poncelet polygons to a special class of operators where we provide simplified geometric proofs of results by Gau, Wang, Wu [GWW16], Spitkovsky and Wegert [WS21], and Tabachnikov and Schwartz [ST16].

1.2 Introduction

Consider the unit circle (i.e., the circle centered at 0 with radius of 1), denoted by $\mathbb{T}$, and an m-sided convex polygon whose vertices lie on the unit circle. We will use the notation m-gon to denote an m-sided polygon. This polygon is said to be **inscribed** in the unit circle (see Figure 5).

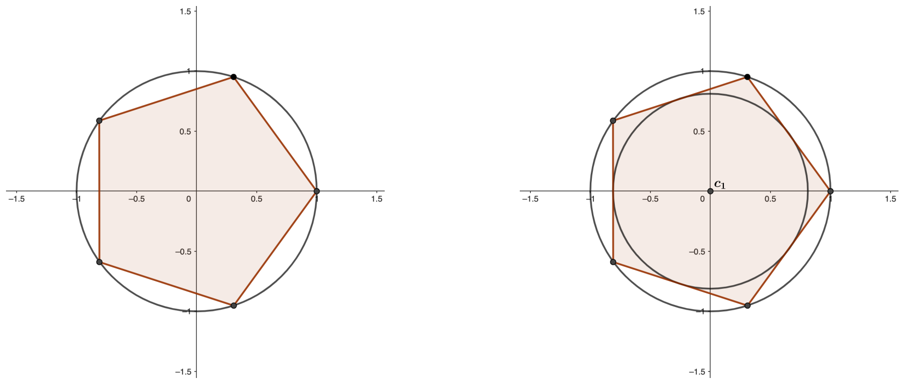

Figure 5: Left:5-gon (pentagon) inscribed in $\mathbb{T}$; Right: Inscribed circle in a 5-gon

Now consider a circle that is tangent to all sides of the m-gon (see the right figure in Figure 5). The polygon is said to **circumscribe** the circle or the circle is **inscribed** in the polygon. It is important to notice that the pair, polygon and circle, must have a specific relation so that the circle is indeed tangent to all the sides of the polygon. But once we find such a pair, it turns out that there are infinitely many m-gons that circumscribe the inner circle and are inscribed in $\mathbb{T}$. This is exactly Poncelet's Theorem:

Theorem 2 (Poncelet's Theorem for Circles) *If there is at least one m-gon that is inscribed in $\mathbb{T}$ and circumscribes an inner circle, then each point on $\mathbb{T}$ is a vertex for such an m-gon.*

If we connect "diagonal" lines that skip over 1 vertex of the polygon, we get a new polygon (or, when $m = 4$, a point). Now we can find another circle that is inscribed in the new polygon (see Figure 7), [Mir12]. Repeating this process by skipping over k vertices for $1 \leq k \leq \frac{m}{2}$ (m even) and $1 \leq k \leq \frac{m-1}{2}$ (m odd), we get a family of circles that are inscribed in polygons. We will see (Proposition 21) that if the first curve that we obtain using this process is a circle, the curves that we obtain from skipping

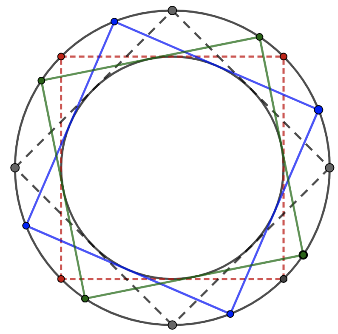

Figure 6: Example of Poncelet's theorem, $m = 4$

points will be too. Note that, for example, when $n = 5$, we can skip 1 point and obtain a curve, skip 2 points to obtain a second curve, but skipping 3 points yields the same curve as skipping 2 points.

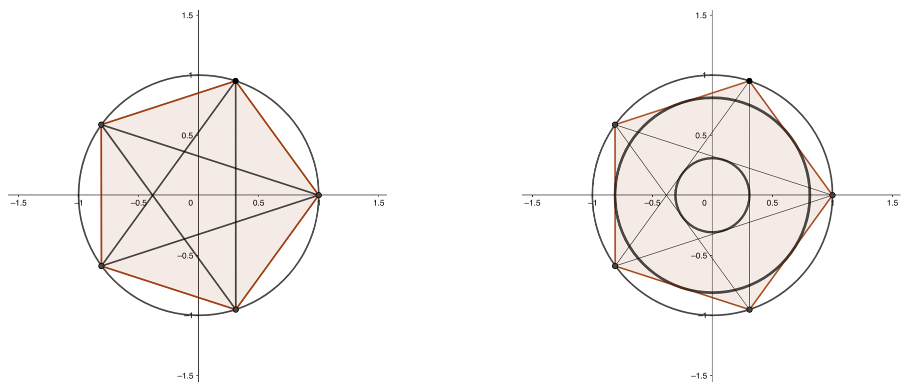

Figure 7: Left: Lines skipping over 1 vertex, $m = 5$; Right: Family of circles, $m = 5$

Our work is focused around one main question: **If you know where one center in the family of circles is, do you know where all of the other circles are centered as well?** This is a natural question; as seen by the examples in the introduction, circles inscribed in polygons appear in numerous examples in the real world.

The question of when the numerical range of an operator is circular is still open [PT02]. In 2016, Gau, Wang, and Wu [GWW16] noted that this general problem is difficult and considered an easier question: *when is the numerical range of an operator* *satisfying a particular condition circular?* They looked at a special class of operators called **partial isometries**, which are operators that preserve distance on a specific subset of the domain. They showed that if the numerical range of a partial isometry on a space of dimension $n \leq 4$ is circular, then it must be centered at the origin. Their proof goes through each dimension $n = 2, 3, 4$ and for each n, they prove this result for all possible dimensions of the kernel of the operator. In Section 3, we simplify and clarify each proof.

In 2021, Spitkovsky and Wegert [WS21] proved the same statement for all n for a certain class of partial isometries whose dimension of the kernel is 1. Their proof involves elliptic integrals and requires a high level of mathematical understanding to follow. Our work re-proves Spitkovsky and Wegert's theorem using information gleaned from the package of Poncelet circles. In Section 3.1, we give a new proof of a result due to Tabachnikov and Schwartz [ST16] that shows that the center of mass of the vertices is constant. In other words, the sum of the vertices of the circumscribing polygons is constant. Then we show that if the dimension of the kernel is one and one of the circles in the package is centered at the origin, all of the circles must be centered at the origin. We do this in two ways. First, we provide a straight forward projective geometric proof in Section 4.2 that follows the work done by Mirman closely, but hides what is happening geometrically. In Section 4 we provide a classical geometric proof of this result. In addition, in Section 3, we provide modified proofs of the cases in which the dimension of the kernel of the operator is greater than one and the operator is defined on an n-dimensional space with $n \leq 4$.

2 Background

A **Hilbert space** H is a complete inner product space such that the norm is defined by $\|f\|^2 = \|f\|_H^2 = \langle f, f\rangle$ for $f \in H$. A **bounded linear operator** A **on a complex Hilbert space** H is a linear transformation $A : H \to H$, for which there exists a real number $M > 0$ such that $\|Ax\|_H \leq M\|x\|_H$ for all $x \in H$. The **numerical range** of an operator A on H, is defined by

$$W(A) = \{\langle Ax, x\rangle : x \in H, \|x\| = 1\}.$$

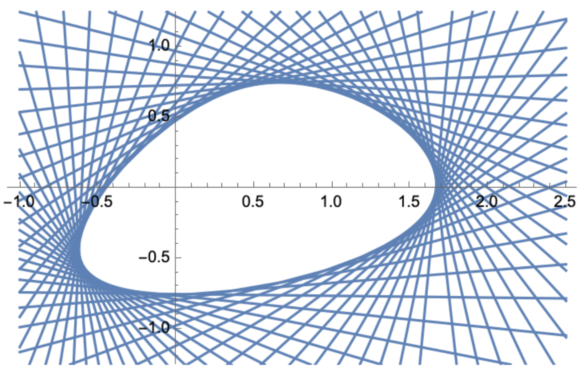

Figure 8: The lines form an envelope of the boundary of $W(A)$

$$A = \begin{pmatrix} 1.5 & 1 & 0 \\ 0 & 0 & 0.5 \\ 0 & i & 0 \end{pmatrix}$$

Figure 9: Matrix A

Figure 10: Figure and Matrix

A famous theorem regarding the numerical range is the Toeplitz-Hausdorff theorem.

Theorem 3 (Toeplitz-Hausdorff Theorem) *The numerical range, $W(A)$, of an arbitrary linear operator A on a Hilbert space (real or complex) is convex.*

When the operator is bounded, the numerical range is a bounded convex subset of H. Let the **spectrum of** A be the set of eigenvalues of A. Then it is also known that

$W(A)$ contains the spectrum of A. Note that when A is a matrix, we are working on a finite dimensional space and $W(A)$ is closed.

A bounded linear operator on H is an **isometry** if $\|Ax\| = \|x\|$ for all $x \in H$; in this case A preserves distance. Denote by $(\ker A)^{\perp}$ the orthogonal complement of the kernel of A. Then a bounded linear operator on H is a **partial isometry** if $\|Ax\| = \|x\|$ for all $x \in (\ker A)^{\perp}$. Partial isometries form an important class in operator theory as they are used in the polar decomposition of arbitrary operators, and they play a key role in the dimension theory of von Neumann algebras. Furthermore, several well-known operators are partial isometries. First, clearly, all isometries are partial isometries. Recall that a unitary operator satisfies $U^*U = UU^* = I$, where U^* denotes the adjoint of U. We also have the following (well-known) result.

Proposition 4 *A unitary operator is a surjective isometry.*

Proof: Let U be a unitary operator. Thus, $\langle U^*Ux, x\rangle = \langle x, x\rangle = \|x\|^2$ for all $x \in H$ and $\langle U^*Ux, x\rangle = \langle Ux, Ux\rangle = \|Ux\|^2$ so $\|Ux\| = \|x\|$. Thus U is an isometry. Since U is invertible, it is also surjective. □

A unitary operator U is also a partial isometry. We also have the following result.

Proposition 5 *Let A be a bounded linear operator. Let P be the orthogonal projection onto $(\ker A)^{\perp}$. Then P is a partial isometry.*

The proof is straightforward. More on partial isometries can be found in [Sko14].

Let I_n denote the $n \times n$ identity matrix. Halmos and McLaughlin [HM63] showed that a partial isometry A can be represented by $\begin{pmatrix} 0 & B \\ 0 & C \end{pmatrix}$ on $H = \ker A \oplus (\ker A)^{\perp}$

where $B^*B + C^*C = I_{\dim\left((\ker A)^\perp\right)}$. Notice that $A^*A = \begin{pmatrix} 0 & 0 \\ B^* & C^* \end{pmatrix} \begin{pmatrix} 0 & B \\ 0 & C \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & B^*B + C^*C \end{pmatrix}$. Since $B^*B + C^*C = I_{\dim\left((\ker A)^\perp\right)}$, we have that A^*A is the identity on $(\ker A)^\perp$.

Proposition 6 *If A is an invertible $n \times n$ matrix that is a partial isometry (i.e., if 0 is not in the spectrum), then A is unitary.*

Proof: Let $H = \mathbb{C}^n$. If A is invertible, then $\ker A = \{0\}$ so $(\ker A)^\perp = H$. Then $\langle A^*Ax, x\rangle = \langle Ax, Ax\rangle = \|Ax\|^2 = \langle x, x\rangle$ for all x, as A preserves length on $(\ker A)^\perp = H$. Thus, $\langle A^*Ax - x, x\rangle = 0$ for all $x \in H$. By [Axl15], we have $A^*A = I$. Then $AA^* = I$ follows. □

Thus, for such an $n \times n$ matrix, the spectrum is contained in the unit circle. The theorem above can be stated in more generality, and it follows that the spectrum of an invertible partial isometry is a non-empty compact subset of the unit circle, $\mathbb{T}$. This property led

Arlen Brown to characterize the spectra of partial isometries in [Bro53].

Theorem 7 *If a nonempty compact subset of the closed unit disk contains the origin, then it is the spectrum of some partial isometry.*

Gau, Wang, and Wu explained why Theorem 7 holds and narrowed this statement to examine partial isometries A whose numerical range is circular, with center c and radius r. Gau, Wang and Wu proved the following theorem.

Theorem 8 ([GWW16]) *Let A be an $n \times n$ partial isometry and $n \leq 4$. If $W(A) = \{z \in \mathbb{C} : |z - c| \leq r\}$ where $r > 0$, then $c = 0$.*

Gau, Wang, and Wu conjectured that this would hold for a general n. In 2021, Spitkovsky and Wegert proved Gau and Wu's conjecture for a special class of partial isometries. This class of partial isometries is associated with a family of functions called **finite Blaschke products.** Let $\mathbb{D} = \{z : |z| < 1\}$ be the open unit disk.

Definition 9 *Let $\lambda_1, \ldots, \lambda_n \in \mathbb{D}$. Then a finite Blaschke of degree n is a function of the form*

$$B(z) := \gamma \prod_{k=1}^{n} \frac{z - \lambda_k}{1 - \overline{\lambda_k} z} \text{ where } |\gamma| = 1.$$

Blaschke products are bounded analytic functions on $\mathbb{D}$ and are widely studied for their nice properties such as their involvement in factorization theorems. Our interest in these functions lies in something known as a Blaschke curve. Recall that a Blaschke product of degree n maps the circle, $\mathbb{T}$, to itself and the open unit disk, $\mathbb{D}$, to itself. It is an n-to-1 map of the unit circle.

Definition 10 *Let B be a Blaschke product of degree n and define $\widehat{B}$ by $\widehat{B}(z) := zB(z)$. For $t \in \mathbb{T}$, let P_t be the convex $(n+1)$-gon with vertices at the preimages $\widehat{B}^{-1}(t)$ of t. Then the curve obtained as the envelope of P_t for $t \in \mathbb{T}$ is called the Blaschke curve.*

In Figure 11, the values of B on $\mathbb{T}$ that have the same argument, mod 2π, are colored with the same color. The Blaschke curve is the envelope of the line segments between points $z, w \in \mathbb{T}$ closest to each other with $B(z) = B(w)$; that is, line segments that join subsequent like colors on $\mathbb{T}$.

A finite Blaschke product B is associated with an operator known as the compression of the shift operator, denoted S_B, acting on a finite-dimensional Hilbert space. Let H^2 denote the Hardy space on $\mathbb{T}$.

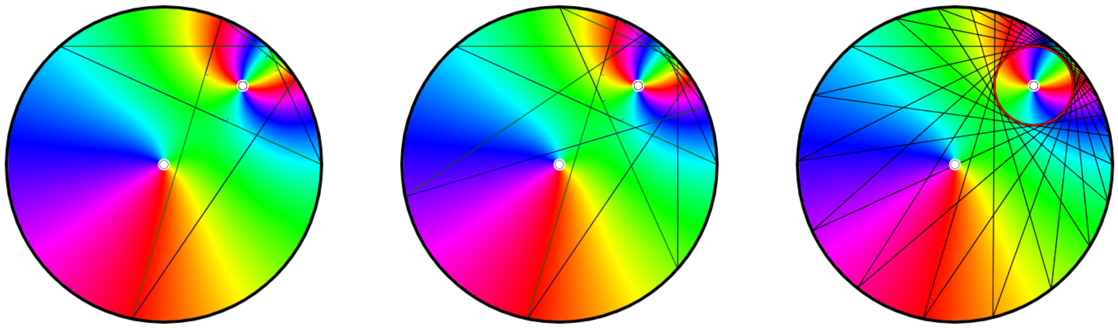

Figure 11: Example of Blaschke curve, $n = 3$

Definition 11 *The Hardy space H^2 is the class of holomorphic functions f on $\mathbb{D}$ satisfying*

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \sum_{n=0}^{\infty} |c_n|^2 < \infty.$$

The model space $K_B = H^2 \ominus BH^2 = \{f \in H^2 : \langle f, Bg \rangle = 0 \text{ for all } g \in H^2\}$. Then our compression of the shift operator $S_B : K_B \to K_B$ is defined by $S_B(f) = P_B(zf)$, where P_B is the orthogonal projection from H^2 onto K_B. A matrix M is of class S_n if M is a contraction (i.e., $\|M\| \leq 1$), the eigenvalues of M are in $\mathbb{D}$, and $\text{rank}(I_n - M^*M) = 1$. It turns out that S_n consists of operators that are unitarily equivalent to S_B with B a finite Blaschke product; thus, a matrix representing a compression of the shift operator with a finite Blaschke product symbol is in the class S_n [DGSV18]. In [GW03], they that show the zeros of B, namely $\lambda_1, \ldots, \lambda_n$, are also the eigenvalues of S_B. Further, the numerical range of S_B is the closure of the interior of the Blaschke curve.

Many properties of the operators in S_n have been established by Gau and Wu, Mirman, and Daepp et.-al ([GW03], [DGV10], [DGSW21], [DGSV18], [DGSV17], [GWW16], [Mir12]). To understand some of these properties from Mirman's perspective, we include a section on **pencils of circles.**

2.1 Pencil of Circles

The main types of pencils of circles are: elliptic pencils, hyperbolic pencils, and parabolic pencils. Two circles determine two pencils: there is a unique pencil that contains the circles and there is a pencil of circles orthogonal to them. The type of pencil that we are interested in here is a **hyperbolic pencil** or a **non-intersecting pencil of circles** (see Figure 12). These circles are determined by two points A and B such that the centers of the circles lie on the line passing through A and B.

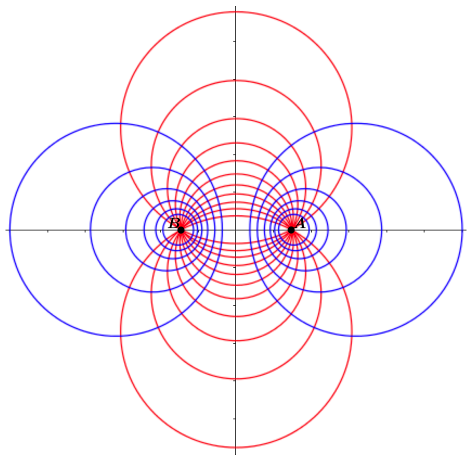

Figure 12: Pencils of circles; blue circles are the hyperbolic pencil [Wik]

The most familiar definition of a circle with center c and radius r is the set of points $X = (x_1, x_2)$ a distance r from $c = (c_1, c_2)$ ($\{X \in \mathbb{R}^2 : \sqrt{(c_1 - x_1)^2 + (c_2 - x_2)^2} = r\}$). But there are equivalent definitions of a circle. Apollonius discovered that a circle could be defined as the set of points X that have a given ratio $d = d_1/d_2$ to two given points (A and B in our case). Thus, each circle in the hyperbolic pencil is associated to a positive number d and is defined to be the set of points such that

$$\{X : \frac{dist(XA)}{dist(XB)} = d\}.$$

Definition 12 *For two circles, C_1 and C_2 with radii r_1 and r_2 and centers c_1 and c_2*

respectively, the power of a point P with respect to C_1 and C_2 is

$$\Pi_1(P) = |Pc_1|^2 - r_1^2 \text{ and } \Pi_2(P) = |Pc_2|^2 - r_2^2.$$

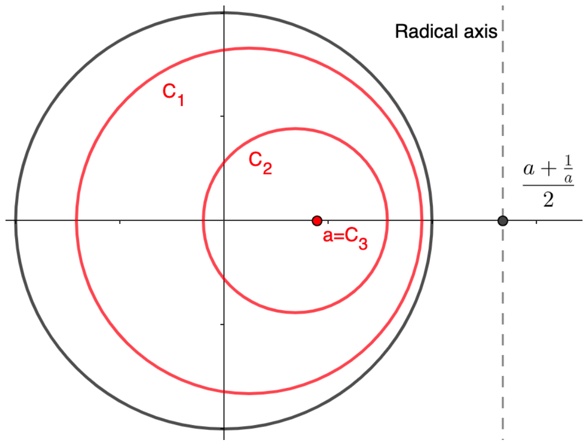

Figure 13: Pencil of circles with the radical axis

Definition 13 *The radical axis is $\{P : \Pi_1(P) = \Pi_2(P)\}$. A pencil of circles is a family of circles that have the same radical axis and the centers of the circles lie on a line.*

The two important facts about pencils of circles that we will use are the following. First, the centers of all of the circles in the pencil are collinear (i.e., lie on one line). And second, any two circles in the pencil share the same **radical axis** and the radical axis is perpendicular to the line segment passing through the centers of the circles in the pencil (Radical Axis Theorem [oPS]).

Hyperbolic pencils of circles appear in real life as well. Suppose you have two parallel wires with current flowing in opposite directions. Each wire will emit a magnetic field whose direction follows the right-hand rule (see Figure 14 where the limiting points act as the wires). These magnetic field lines form circles that resemble a pencil of hyperbolic circles.

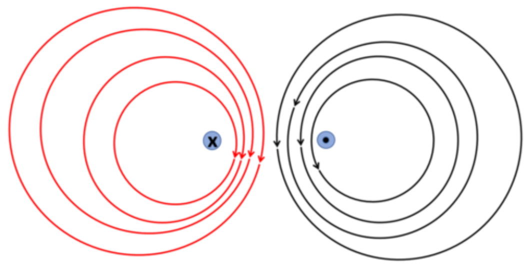

Figure 14: Magnetic field lines of parallel wires

The family of all circles through the points A and B (including the line AB) is the **elliptic pencil** of circles through A and B. The family of all **Apollonian circles** (including the perpendicular bisector of AB) is the hyperbolic pencil defined by A and B. The set of circles orthogonal to the circles of an elliptic pencil is a hyperbolic pencil, and conversely.

The points A and B are called the *limiting points* of the hyperbolic pencil [Pam]. We can get a formula for the limiting points given two circles in the pencil of radii r and R with centers separated by a distance d. Consider the unit circle $\mathbb{T}$ and the circle with radius R centered at $(c, 0)$ with $0 < c < 1$. Then [Wei] shows the limiting points are as follows:

$$A = \frac{c^2 - R^2 + 1 + \sqrt{(c^2 - R^2 + 1)^2 - 4c^2}}{2c}$$

and

$$B = \frac{c^2 - R^2 + 1 - \sqrt{(c^2 - R^2 + 1)^2 - 4c^2}}{2c}.$$

Given the equations above, the points A and B also satisfy the following:

$$A + B = A + \frac{1}{A} = \frac{1 + c^2 - R^2}{c}.$$

The limiting points in Figure 12 are the common intersection points of the red circles.

Because one of the centers of our circles is zero and all the centers lie on a line, without loss of generality, we may assume that the centers of the circles are real. In [Mir12], Mirman gives an invariant, I_C, of the pencil of circles given the center $c > 0$ and radius R of the first circle in $\mathbb{D}$ in the Poncelet package:

$$I_C = \frac{1 + c^2 - R^2}{2c}.$$

In other words, every circle in the package with center c_k and radius R_k satisfies the following relation:

$$\frac{1 + c_k^2 - R_k^2}{2c_k} = \frac{1 + c^2 - R^2}{2c}.$$

But if one of the circles degenerates to a point (see the proof of Theorem 8, case $n = 3$), then $R_k = 0$ so

$$\frac{1 + c_k^2}{2c_k} = \frac{1 + c^2 - R^2}{2c}.$$

Recall that each Euclidean disk with $c_k \neq 0$, $D_E(c_k, R_k)$, is also a pseudohyperbolic disk, where the pseudohyperbolic disk has (pseudohyperbolic) center z_0 and radius r_k, with

$$|z_0| + \frac{1}{|z_0|} = \frac{1 + c_k^2 - R_k^2}{c_k} = \frac{1 + c^2 - R^2}{c}$$

and

$$c_k = z_0(1 - r_k R_k),$$

so c and z_0 have the same sign. For the unit circle, we may think of every $z_0 \in \mathbb{D}$ as a pseudohyperbolic center by taking $r = 1$. In this case, $r = R = 1$ and $c = 0$.

Remark 14 *All of the circles in the pencil have the same pseudohyperbolic center,* z_0.

2.2 Poncelet's Theorem

We return to Poncelet's closure theorem. After Poncelet first published his results, he then generalized them to what we now call Poncelet's general theorem.

Theorem 15 (Poncelet's general theorem) *Let C_1 be a circle and let non-intersecting circles $a_1, a_2, \ldots, a_n$ be inside C_1 belonging to the same pencil. Starting at an arbitrary point $w_1 \in C_1$ construct points $w_2, \ldots, w_{n+1} \in C_1$ so that $\overline{w_j w_{j+1}}$ is tangent to a_j. If $w_n = w_1$, then we always get back to the starting point in the nth step, no matter where we start.*

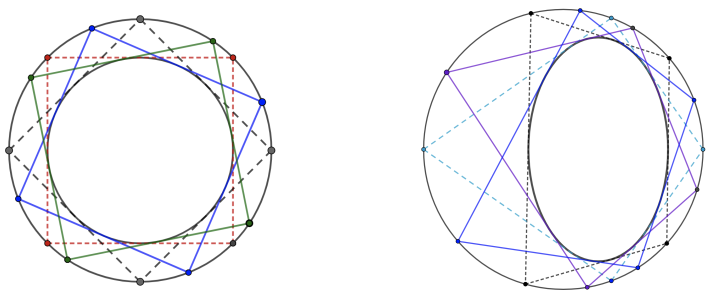

Figure 15: Examples of Poncelet's theorem, $m = 4$.

For $t \in \mathbb{T}$, the convex $(n+1)$-gon with vertices at the preimages $\widehat{B}^{-1}(t)$ of t is denoted by P_t and $\text{conv}(P_t)$ denotes the convex hull of the P_t. Gau and Wu proved the following:

Theorem 16 ([GWW16]) *Let B be a Blaschke product of degree n and define $\widehat{B}$ by $\widehat{B}(z) := zB(z)$. Then the numerical range of S_B is*

$$W(S_B) = \bigcap_{t \in \mathbb{T}} conv(P_t).$$

In view of the fact that the curve bounding the numerical range is inscribed in a polygon P_t for each $t \in \mathbb{T}$, such a curve is called a *Poncelet curve*. Spitkovsky and Wegert showed

Theorem 17 ([WS21]) *Let $\widehat{B}$ be a Blaschke product of degree $n+1$ where $n \geq 2$, $\widehat{B}(0) = 0$ and $\widehat{B}'(0) = 0$. If the Blaschke curve associated with $\widehat{B}$ is a circle, then $\widehat{B}(z) = z^{n+1}$.*

We will provide another, more elementary, proof of this theorem in Section 4.

Another way to approach the proof of this theorem is by inspecting something called a **package of Poncelet circles**.

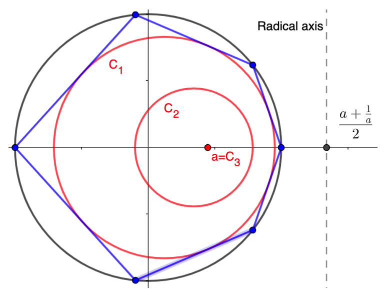

Figure 16: Package of Poncelet circles, $m = 6$

Definition 18 *Let $t \in \mathbb{T}$ and let P_t denote the convex $(n+1)$-gon whose vertices are $\widehat{B}^{-1}(t)$, ordered by principal argument. A package of Poncelet curves is the collection of curves created by joining line segments skipping over $k = 1, 2, \ldots, \lfloor \frac{n+1}{2} \rfloor$ vertices of P_t. Each Poncelet curve C_k is tangent to the chords that skip over k vertices.*

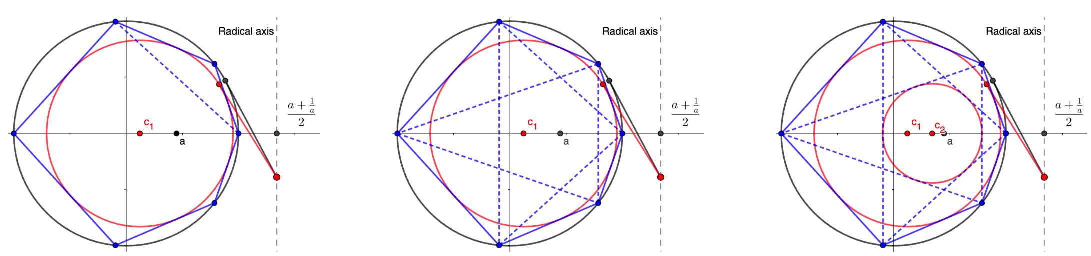

Figure 17: Method to get C_2 in the package

We now introduce two other versions of Poncelet's theorem to show why all the curves in the package are circles.

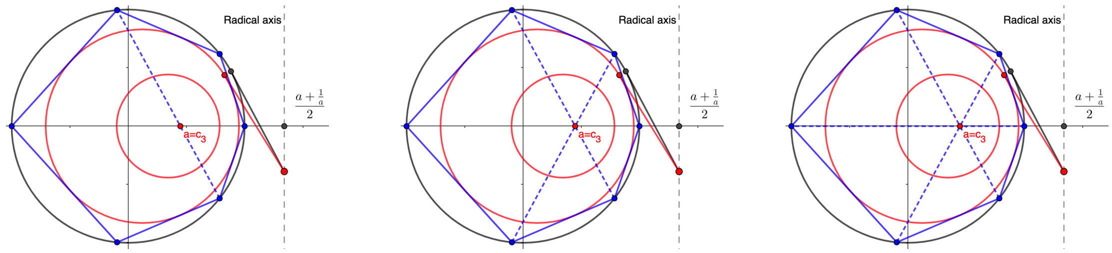

Figure 18: Method to get C_3 in the package

Theorem 19 ([Ber10], p. 218) *Consider the pencil of curves defined by the pair of circles C, C_1. Let C_2 be an arbitrary circle (interior to C) belonging to the pencil. We traverse all these circles in the same direction, and with each $m \in C$ we associate the point n where the tangent from m to C_1 encounters C again; then from n we follow the tangent to C_2 that cuts C again at p. Then the line mp, when m traverses C, envelops a circle C_3 belonging to the pencil considered.*

Theorem 20 ([HMFPS22] Theorem B) *Let C be a closed convex curve in $\mathbb{D}$ and suppose that there is an n-sided polygon P_0 inscribed in $\mathbb{T}$ and circumscribed about C. If the curve C is a connected component of a real algebraic curve Γ in $\mathbb{D}$ of class $n-1$ such that each diagonal of P_0 is tangent to Γ, then for every point $z \in \mathbb{T}$, there exists an n-sided polygon $P(z)$ inscribed in $\mathbb{T}$ and circumscribed about C such that z is a vertex of $P(z)$ and each diagonal of $P(z)$ is tangent to Γ. In the special case when C is an ellipse, there always exists such an algebraic curve Γ and this curve decomposes into $(n-1)/2$ ellipses if n is odd, and $(n-2)/2$ ellipses and an isolated point if n is even.*

Proposition 21 *Consider the package of Poncelet curves determined by $\mathbb{T}$ and the largest circle C_1 in the package contained in $\mathbb{D}$. Then every curve in the package of Poncelet curves must be a circle.*

Proof: Let A be the matrix representing the operator S_B, where B is a finite Blaschke product, and the boundary of $W(A)$ is C_1.

We will use the construction in Theorem 19 to get all the circles in the pencil defined by $\mathbb{T}$ and C_1 using the following method.

To find C_2, we take two neighboring line segments tangent to C_1, mn and np, and consider the line mp. Notice that mp skips over 1 vertex of our largest polygon P_1, which is tangent to C_1. Then by Theorem 19, mp, as m traverses $\mathbb{T}$, envelopes C_2 in the pencil.

To find C_3, we take a line segment tangent to C_1, mn, and a line segment tangent to C_2, np, and consider the line mp. Since mn skips over 0 vertices and np skips over 1 vertex, mp skips over $0 + 1 + 1 = 2$ vertices of our largest polygon P_1, where the extra 1 vertex is the vertex at the point n. Then mp, as m traverses $\mathbb{T}$, envelopes C_3 in the pencil.

We repeat the process and have included a general formula to get C_k.

Case 1: $k \in 2\mathbb{N}$. To get C_k, we take two neighboring line segments tangent to $C_{\frac{k}{2}}$, mn and np, and consider the line mp. Since mn and np skips over $\frac{k}{2} - 1$ vertices, mp skips over $\frac{k}{2} - 1 + \frac{k}{2} - 1 + 1 = k - 1$ vertices of our largest polygon P_1. Then mp, as m traverses $\mathbb{T}$, envelopes C_k in the pencil.

Case 2: $k \in 2\mathbb{N}+1$. To get C_k, we take a line segment tangent to $C_{\frac{k-1}{2}}$, mn, and a line segment tangent to $C_{\frac{k+1}{2}}$, np, and consider the line mp. Since mn skips over $\frac{k-1}{2} - 1$ vertices and np skips over $\frac{k+1}{2} - 1$ vertex, mp skips over $\frac{k-1}{2} - 1 + \frac{k+1}{2} - 1 + 1 = k - 1$ vertices of our largest polygon P_1. Then mp, as m traverses $\mathbb{T}$, envelopes C_k in the pencil.

Since this construction is the process to obtain Mirman's vertices of the package of curves, we have that if the largest curve in $\mathbb{D}$ in the package is circular, then all of the curves must be circles as well.

□

We know the zeros of the Blaschke product B are the eigenvalues of S_B ([GMR18] Corr.12.6.7). We now obtain more information about the zeros of B; that is, the eigenvalues of S_B.

Consider $P(z,w) = P(z,w;a_1,\ldots,a_{N-1})$, where

$$P(z,w) = \frac{1}{w-z}\left(w\prod_{k=1}^{N-1}(w-a_k)(1-\overline{a_k}z) - z\prod_{k=1}^{N-1}(z-a_k)(1-\overline{a_k}w)\right).$$

Mirman states [Mir05, Section 2] that the zeros of P (namely, $a_1,\ldots,a_{N-1}$) are the foci for ellipses in the package. Thus, they will give us the centers of our circles. Notice that if $P(z,w)=0$, and $a_1,\ldots,a_{N-1}$ are the zeros of B, then

$$\widehat{B}(w) = w\prod_{k=1}^{N-1}\frac{w-a_k}{1-\overline{a_k}w} = z\prod_{k=1}^{N-1}\frac{z-a_k}{1-\overline{a_k}z} = \widehat{B}(z). \tag{1}$$

Recall that our Poncelet polygons circumscribing $W(S_B)$ were constructed by connecting the points in $\widehat{B}^{-1}(t)$ (ordered by principal argument) and we see in (1) above that since $t\in\mathbb{T}$ we have $z,w\in\mathbb{T}$ and $z,w\in\widehat{B}^{-1}(t)$.

Spitkovsky and Wegert used elliptic integrals to prove Theorem 17. We re-prove Theorem 17 in Section 4 using elementary geometry, classical projective geometry, and linear algebra. Further, we showed in Proposition 21 that all of the circles in the package belong to one pencil. Then since the centers are colinear and we are assuming one circle in the package is centered at 0, then we can rotate our package of circles such that all of the centers are real. Spitkovsky and Wegert's result follows from Mirman's observation of the invariance of the pseudohyperbolic centers of the circles in the package, but we will give a different proof that demonstrates geometric properties of the circles and circumscribing polygons.

2.3 Kippenhahn Curve

Rudolf Kippenhahn showed that the numerical range is the convex hull of a certain algebraic curve now called the **Kippenhahn curve**. In this section, we will provide a brief discussion of the Kippenhahn curve, as it plays a key role in the proofs in Section 3.

For an $n \times n$ matrix A, let $\Re A = \frac{A+A^*}{2}$ and $\Im A = \frac{A-A^*}{2i}$ denote the real and imaginary parts of the matrix A, respectively. Let I_n be the $n \times n$ identity matrix. Then consider the homogeneous degree-n polynomial $P_A(x,y,z) = \det(x\Re A + y\Im A + zI_n)$. Let $\mathbb{CP}^2$ be the complex projective plane consisting of all equivalence classes $[x;y;z]$ of ordered triples of complex numbers x, y and z that are not all zero. Two triples $[x;y;z]$ and $[x';y';z']$ are equivalent if $x = \lambda x'$, $y = \lambda y'$, and $z = \lambda z'$ for some nonzero λ. Now let $C(A)$ be the algebraic curve that is dual to the algebraic curve determined by $P_A(x,y,z) = 0$ in the complex projective plane $\mathbb{CP}^2$. In other words, $C(A) = \{(u,v,w) \in \mathbb{CP}^2 : ux + vy + wz = 0 \text{ is tangent to } P_A(x,y,z) = 0\}$. Kippenhahn showed that the numerical range $W(A)$ is the convex hull of the real points of $C(A)$ [Kip51]. Let the Kippenhahn curve of A be the set $C_R(A) = \{a+bi \in \mathbb{C} : a, b \in \mathbb{R} \text{ and } ax + by + z = 0 \text{ is tangent to } P_A(x,y,z) = 0\}$. Then $W(A) = \operatorname{conv}(C_R(A))$. With this result, we will be able to say things about the numerical range using the Kippenhahn curve.

3 Proof of Theorem 8

3.1 Preliminaries

Gau, Wang, and Wu reduced the question to the study of irreducible operators.

Definition 22 *A operator A on H is* ***irreducible*** *if there are no other reducing subspaces (a subspace S such that $A(S) \subseteq S$ and the adjoint A^* satisfies $A^*(S) \subseteq S$) other than the trivial ones $\{0\}$ and $\{H\}$.*

We will assume all of our matrices are irreducible. In the proof of Theorem 8, we also assume $n \leq 4$. For a discussion on why we can assume irreducibility, see [GWW16]. To prove Theorem 8, we must first introduce several important theorems and propositions, the first of which is the famous Elliptical Range Theorem.

Theorem 23 (Elliptical Range Theorem) *Let A be a 2×2 matrix with eigenvalues λ_1 and λ_2. The numerical range of A is an elliptical disk with λ_1 and λ_2 as foci, and $(tr(A^*A) - |\lambda_1|^2 - |\lambda_2|^2)^{1/2}$ as the length of the minor axis.*

There are many proofs of this theorem (see, for example, [PRW21] [Li96]) and it has been a critical result in the study of numerical ranges. This theorem will be particularly useful in proving Theorem 8 for $n = 2$.

We will also use Schur's theorem, which states:

Theorem 24 *If A is an $n \times n$ square matrix with complex entries, then A can be expressed as $A = QUQ^*$ where Q is a unitary matrix, and U is an upper triangular matrix.*

Schur's theorem and a proof of it can be found in Theorem 6.14 of [FIS97].

We continue with an important connection between the eigenvalues and the vertices of the circumscribing Poncelet polygons. First, we must define a **unitary dilation.**

Definition 25 *An operator $U : H' \to H'$ is a unitary dilation of $A : H \to H$ where $H \subset H'$ if U is a unitary operator and $P_H U|_H = A$, where P_H is the orthogonal projection on H.*

Now we introduce the **Takenaka-Malmquist basis** for H, which is used to get a matrix representation of S_B.

Definition 26 *Let us write $\phi_a(z) = \frac{z-a}{1-\bar{a}z}$ where $a \in \mathbb{D}$. For a Blaschke product of degree n with zeros $a_1, \ldots, a_n$, then $B = \mu \prod_{j=1}^{n} \phi_{a_j}$ where $\mu \in \mathbb{T}$. Assume $a_1, \ldots, a_n$ are distinct. Let $\tilde{k}_a(z) = \frac{\sqrt{1-|a|^2}}{1-\bar{a}z}$ be the normalized reproducing kernel for H^2 at the point $a \in \mathbb{D}$. Then the Takenaka-Malmquist basis for K_B is given by*

$$\left(\tilde{k}_{a_1} \prod_{j=2}^{n} \phi_{a_j}, \tilde{k}_{a_2} \prod_{j=3}^{n} \phi_{a_j}, \ldots, \tilde{k}_{a_{n-1}} \phi_{a_n}, \tilde{k}_{a_n} \right).$$

Let A_B be the matrix representing S_B with respect to the **Takenaka-Malmquist basis**. If $a_1, \ldots, a_n \in \mathbb{D}$ are the zeros of B, then A_B has the form:

$$a_{ij} = \begin{cases} a_j & \text{if } i = j \\ \left(\prod_{k=i+1}^{j-1} (-\overline{a_k}) \right) \sqrt{1-|a_i|^2} \sqrt{1-|a_j|^2} & \text{if } i < j \\ 0 & \text{if } i > j \end{cases}.$$

This representation is also valid when the zeros are not distinct. For each $\lambda \in \mathbb{T}$, the unitary dilation U_λ is defined as follows:

$$a_{ij} = \begin{cases} a_j & \text{if } 1 \le i, j \le n \\ \lambda \left(\prod_{k=1}^{j-1}(-\overline{a_k})\right) \sqrt{1-|a_j|^2} & \text{if } i = n+1 \text{ and } 1 \le j \le n, \\ \left(\prod_{k=i+1}^{n}(-\overline{a_k})\right) \sqrt{1-|a_i|^2} & \text{if } j = n+1 \text{ and } 1 \le i \le n, \\ \lambda \prod_{k=1}^{n}(-\overline{a_k}) & \text{if } i = j = n+1 \end{cases}. \tag{2}$$

Up to unitary equivalence, this is the complete set of unitary 1-dilations of A_B. Using this, we obtain the next theorem, which shows that the average of the vertices of all circumscribing polygons lie on a circle. Further, if one of the eigenvalues of A_B is zero, then this circle degenerates to a point and the sum of the vertices of each circumscribing polygon is the same. For each $\lambda \in \mathbb{T}$, there is a unique unitary dilation of A_B, defined by U_λ (up to unitary equivalence) as defined in equation (2).

Theorem 27 *If $a_1, \dots, a_n$ are the eigenvalues of A_B and $\lambda_1, \dots, \lambda_{n+1} \in \mathbb{T}$ are the eigenvalues of a unitary 1-dilation of A_B; that is, a matrix of the form U_λ with $\lambda \in \mathbb{T}$, then*

$$\sum_{j=1}^{n+1} \lambda_j = tr(U_\lambda) = \sum_{j=1}^{n} a_j + \lambda \prod_{j=1}^{n} (-\overline{a_j}).$$

Consequently, the matrix A_B has 0 as an eigenvalue if and only if the unitary dilations U_λ, $\lambda \in \mathbb{T}$ have the same trace.

Proof: The matrix U_λ is a unitary 1-dilation of the matrix A_B; that is, we obtain U_λ from A_B by adding one row and one column to the matrix A_B. Therefore, the trace of U_λ is the sum of the diagonal entries of A_B plus the entry in the final row and column, $\lambda \prod_{j=1}^{n}(-\bar{a}_j)$. So, $\text{tr}(U_\lambda) = \sum_{j=1}^{n} a_j + \lambda \prod_{j=1}^{n}(-\overline{a_j})$. But U_λ is unitary and therefore unitarily equivalent to a diagonal matrix with the eigenvalues $\lambda_1, \dots, \lambda_{n+1}$

on the diagonal. Since the trace is a unitary invariant, $\mathrm{tr}(U_\lambda) = \sum_{j=1}^{n+1} \lambda_j$. Thus,

$$\sum_{j=1}^{n} a_j + \lambda \prod_{j=1}^{n} (-\overline{a_j}) = \mathrm{tr}(U_\lambda) = \sum_{j=1}^{n+1} \lambda_j.$$

□

This result is due to Tabachnikov and Schwartz and a different proof can be found in [WS21].

Remark 28 *Note that the unitary dilations give the vertices of the circumscribing polygons. Thus, when one of the eigenvalues of A_B is zero, the sum of all the vertices of every circumscribing polygon is the same.*

In [Gau06], they show that for a 4×4 matrix whose numerical range is an elliptical disk, $C_R(A)$ has a factor of order 2. By duality, it follows that the homogeneous polynomial P_A also has a factor of degree 2. Note that P_A is of degree 4.

Remark 29 ([Gau06] pp. 118) *Let A be a 4×4 partial isometry. If $W(A)$ is an elliptic disk, then P_A can be decomposed either by two factors of degree 2 or by one factor of degree 2 and two factors of degree 1. So there are two cases: when the Kippenhahn curve of a 4×4 matrix consists of the boundary of an elliptical disk and two points as well as when the Kippenhahn curve consists of the boundary of two elliptical disks.*

Gau provided conditions for the various possibilities for the Kippenhahn curve of a 4×4 matrix, which we have included below as Theorem 30 and Corollary 31.

Theorem 30 (Theorem 3 [Gau06]) *Let A be a 4×4 matrix of the form* $A = \begin{bmatrix} \lambda_1 & a & d & f \\ 0 & \lambda_2 & b & e \\ 0 & 0 & \lambda_3 & c \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}$. *Then $C_R(A)$ consists of two points and one ellipse if and only if*

1. $r^2 = |a|^2 + |b|^2 + |c|^2 + |d|^2 + |e|^2 + |f|^2$,

2. $r^2\lambda_i\lambda_j = |a|^2\lambda_3\lambda_4 + |c|^2\lambda_1\lambda_2 + |d|^2\lambda_2\lambda_4 + |e|^2\lambda_1\lambda_3 + |f|^2\lambda_2\lambda_3 - (\lambda_1 bc\overline{e} + \lambda_2 cd\overline{f} + \lambda_3 ae\overline{f} + \lambda_3 ab\overline{d}) + abc\overline{f}$,

3. $r^2(\lambda_i + \lambda_j) = (|b|^2 + |c|^2 + |e|^2)\lambda_1 + (|c|^2 + |d|^2 + |f|^2)\lambda_2 + (|a|^2 + |e|^2 + |f|^2)\lambda_3 + (|a|^2 + |b|^2 + |d|^2)\lambda_4 - (bc\overline{e} + cd\overline{f} + ae\overline{f} + ab\overline{d})$, *and*

4. $r^2\alpha_i\alpha_j = |a|^2\alpha_3\alpha_4 + |b|^2\alpha_1\alpha_4 + |c|^2\alpha_1\alpha_2 + |d|^2\alpha_2\alpha_4 + |e|^2\alpha_1\alpha_3 + |f|^2\alpha_2\alpha_3 - (\alpha_1\Re(bc\overline{e}) + \alpha_2\Re(cd\overline{f}) + \alpha_3\Re(ae\overline{e}) + \alpha_4\Re(ab\overline{d}) - \frac{1}{4}(|a|^2|c|^2 + |d|^2|e|^2 + |b|^2|f|^2 - 2\Re(a\overline{c}\overline{d}e) - 2\Re(b\overline{d}\overline{e}f) - 2\Re(abc\overline{f}))$

When the Kippenhahn curve consists of two ellipses, we will use the following corollary.

Corollary 31 (Corollary 6 [Gau06]) *Let A be a 4×4 matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. Then $C_R(A)$ consists of two ellipses, one with foci λ_k, λ_l and minor axis of length r, the other with foci λ_i, λ_j and minor axis of length s if and only if:*

1. $r^2 + s^2 = tr(A^*A) - \sum_{i=1}^{4} |\lambda_i|^2 = \gamma^2$,

2. $r^2\lambda_i\lambda_j + s^2\lambda_k\lambda_l = \sum_{1 \leq n < m \leq 4}(\gamma^2 + |\lambda_n|^2 + |\lambda_m|^2)\lambda_n\lambda_m + tr(A^*A^3) - tr(A)tr(A^*A^2)$,

3. $r^2(\lambda_i + \lambda_j) + s^2(\lambda_k + \lambda_l) = \gamma^2 tr(A) - tr(A^*A^2) + \sum_{n=1}^{4} |\lambda_n|^2\lambda_n$, *and*

4. $r^2\alpha_i\alpha_j + s^2\alpha_k\alpha_l - \frac{1}{4}r^2s^2 = 4\alpha_1\alpha_2\alpha_3\alpha_4 - 4\det ReA$ *where* $\alpha_i = Re(\lambda_i)$.

We are interested in applying these condition to certain partial isometries. In particular, we will apply it in the case when the dimension of the kernel of a 4×4 partial isometry is 2.

As previously mentioned, a consequence of the Halmos and McLaughlin result [HM63] shows that a partial isometry for which the kernel has dimension 2 can be written as

$$A = \begin{bmatrix} 0_2 & B \\ 0_2 & C \end{bmatrix} \text{ where } B = \begin{bmatrix} c & d \\ e & f \end{bmatrix}, C = \begin{bmatrix} \alpha & b \\ 0 & \alpha \end{bmatrix}, \text{ and } B^*B + C^*C = I. \quad (3)$$

The first condition of Corollary 31 gives the following lemma.

Lemma 32 *Let A be a partial isometry. If A is of the form (3), then the minor axis of the Kippenhahn curve is given by $r^2 = 2(1-\alpha^2)$.*

Proof: Let A be a partial isometry of the form (3); that is, $A = \begin{bmatrix} 0_2 & B \\ 0_2 & C \end{bmatrix}$ where $B = \begin{bmatrix} c & d \\ e & f \end{bmatrix}$ and $C = \begin{bmatrix} \alpha & b \\ 0 & \alpha \end{bmatrix}$ satisfy $B^*B + C^*C = I_2$. Then

$$A^*A = \begin{bmatrix} 0_2 & 0_2 \\ 0_2 & I_2 \end{bmatrix}$$

so $tr(A^*A) = 2$. Further, $\sum_{i=1}^{4} |\lambda_i|^2 = \alpha^2$ where λ_i, $1 \leq i \leq 4$ are the eigenvalues of A and $\alpha \in \mathbb{R}$. By condition 1 of Corollary 31, we have $r^2 + \alpha^2 = 2 - 2\alpha^2$. Thus, $r^2 = 2(1-\alpha^2)$. $\square$

Recall that a matrix M is of class S_n if M is a contraction (i.e., $\|M\| \leq 1$), the eigenvalues of M are in $\mathbb{D}$, and $\text{rank}(I_n - M^*M) = 1$. In the proof of Theorem 8, we will use an important proposition by Gau and Wu.

Proposition 33 ([GWW16] Prop. 2.3) *Let A be an $n \times n$ matrix. Then A is an irreducible partial isometry with* $\dim \ker A = 1$ *if and only if A is of class S_n with 0 in $\sigma(A)$.*

This proposition allows us to reduce the case when $\dim \ker A = 1$ to the numerical ranges of compressions of the shift operator for which the Blaschke symbol has a zero at zero. The Blaschke symbol then allows us to use elementary geometry to prove Theorem 8 when $\dim \ker A = 1$.

Lastly, we introduce an important theorem relating the geometry of the numerical range and the eigenvalues.

Theorem 34 *([WG21]) If A is a matrix with numerical range a circular disk, then two of the eigenvalues of A are equal to the center of the disk.*

3.2 Proof of Theorem 8

We turn to the proof of Theorem 8 and we also show that every circle in the Poncelet package is centered at 0. Following Gau, Wang, Wu, we break the proof into cases, depending on the size of the kernel of the matrix A. However, we were able to simplify several of the cases. For instance, when the dimension of the kernel is 1, we know that 0 is an eigenvalue of A so we can apply Proposition 33. Thus, the work in this section provides a simpler proof of Theorem 8 and Theorem 17 for $n = 2, 3, 4$. In Section 4, we provide a complete proof of Spitkovsky and Wegert's result (Theorem 17).

Proof: We consider partial isometries on spaces of dimension $n = 2, n = 3$, and $n = 4$ individually.

Case 1: $n = 2$. Let A be a 2×2 matrix that is a partial isometry with circular numerical range. Using Theorem 23 we know that $W(A)$ must be an elliptical disk. If no focus is 0, no eigenvalue is 0 and we know from Proposition 6 that A is unitary. Therefore, the numerical range is the convex hull of the eigenvalues. If the eigenvalues are distinct, the numerical range of A is not circular. If the eigenvalues are the same, the matrix is a scalar multiple of the identity I_2 and, consequently, reducible. Therefore, the ellipse must have at least one focus at 0. Since we are assuming $W(A)$ is circular, the other focus is $\lambda = 0$, and $W(A)$ is centered at 0.

Case 2: $n = 3$. Let A be a 3×3 partial isometry with circular numerical range. We consider three cases:

- Assume $\dim \ker A = 0$. This case is not possible as then $(\ker A)^{\perp} = H$ so, as above, A is unitary. In this case, $W(A)$ is the convex hull of the finitely many eigenvalues of U, all of which lie on $\mathbb{T}$. Thus, $W(A)$ is not circular.

- Assume $\dim \ker A = 1$. In this case, 0 is an eigenvalue and we know that the eigenvalues are the foci of the curves in the package. We know that the numerical range is circular and, by Proposition 33, A represents a compression S_B, with B a degree-3 Blaschke product. By [GWW16], this circle is a Poncelet circle. Therefore, there are two circles in the package and one of the two circles is centered at 0. Then by Theorem 16, the circumscribing quadrilaterals have vertices determined by the pre-images $\widehat{B}^{-1}(t)$ for $t \in \mathbb{T}$.

  Since there are four vertices in the circumscribing quadrilateral, there is a degenerate circle in the package of circles given by Theorem 20. So the package determined by $\mathbb{T}$ and a circle C_1 inscribed in quadrilaterals consists of C_1 and the degenerate circle, C_2, at the intersection of the diagonals. We show that both C_1 and C_2 are centered at 0. If C_1 is centered at 0, then the vertices of the

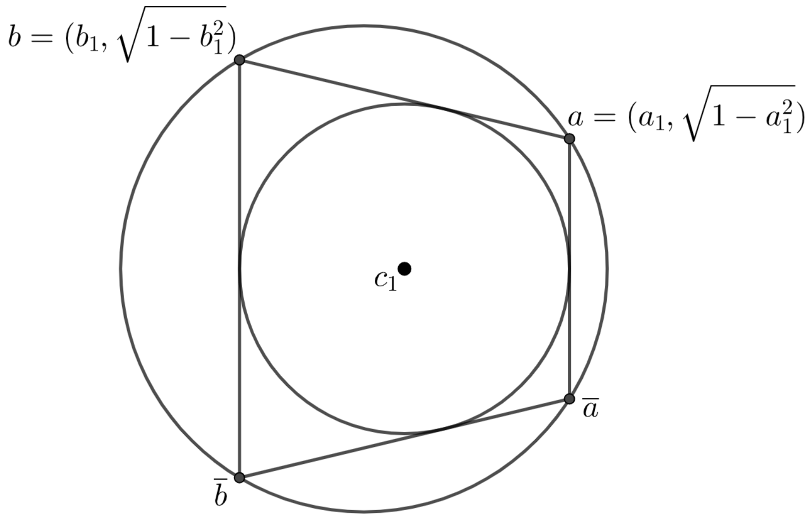

Figure 19: Illustration of the proof when $n = 3, \dim\ker A = 1$

circumscribing quadrilaterals must be evenly spaced. Then the diagonals must pass through 0. Thus, this completes the proof in this case.

Now assume the degenerate circle is 0. Consider the quadrilateral P circumscribing the circle C_1 where one side is vertical. By symmetry, the opposing side must also be vertical. Let us denote the vertices of P, which lie on the unit circle, by $a = (a_1, \sqrt{1-a_1^2})$ and $b = (b_1, \sqrt{1-b_1^2})$, and their conjugates, where $a_1 > 0$ and $b_1 < 0$. Since $0 \in W(A)$ and operators in S_n cannot have an eigenvalue on the boundary, this is possible.

Note that we do not yet know that $a = -b$. The equation of the line passing through a and 0 must be of the form

$$y = \frac{\sqrt{1-a_1^2}}{a_1}x.$$

Notice that $\bar{b} = (b_1, -\sqrt{1-b_1^2})$ must be on this line by the definition of the diagonal of the quadrilateral. So the following must hold:

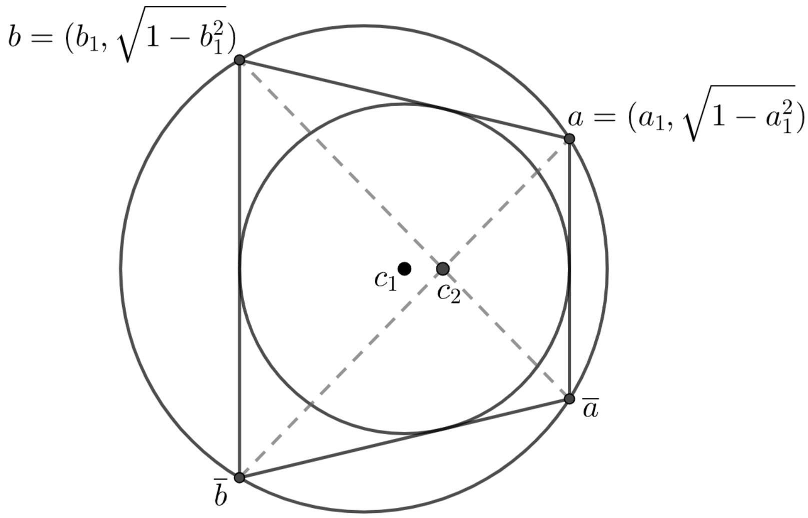

Figure 20: Illustration of the proof when $n = 3, \dim\ker A = 1$; C_2 centered at 0

$$-\sqrt{1-b_1^2} = \frac{\sqrt{1-a_1^2}}{a_1} b_1. \tag{4}$$

Simplifying (4), we get

$$a_1^2 - (a_1 b_1)^2 = b_1^2 - (a_1 b_1)^2$$

so $a_1 = \pm b_1$. Since we know $a_1 > 0$ and $b_1 < 0$, it follows that $a_1 = -b_1$. Thus, C_1 is centered at $c_1 = \frac{a_1+b_1}{2} = 0$.

- Assume $\dim\ker A = 2$.

  Using Theorem 24, we can write A as an upper triangular matrix.

$$\begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & a \end{bmatrix}$$

Since we know that the dimension of the kernel is 2, we can find an orthonormal basis $\{f_1, f_2\}$ for $\ker A$. Let f_3 be a unit vector orthogonal to $\{f_1, f_2\}$, so that $\{f_3\}$ is a basis for $(\ker A)^{\perp}$. Then $\{f_1, f_2, f_3\}$ form a basis for the vector space $\mathbb{C}^3$, as they are linearly independent and span $\mathbb{C}^3$. With respect to this basis we get a matrix representation

$$\begin{bmatrix} 0 & 0 & y_1 \\ 0 & 0 & z_1 \\ 0 & 0 & a \end{bmatrix}.$$

Now using Theorem 34, we know that two of the eigenvalues must be the same and will also be the center of $W(A)$. If the two eigenvalues are 0 and 0, we are done. If the two eigenvalues are 0 and a, then $a = 0$ and so $W(A)$ is centered at 0.

Case 3: $n = 4$. Let A be a 4×4 partial isometry with circular numerical range. We consider four cases.

- Assume $\dim \ker A = 0$. Then A is unitary, so $W(A)$ is the convex hull of the eigenvalues of A, all of which lie on $\mathbb{T}$. Therefore, $W(A)$ is not circular.

- Assume $\dim \ker A = 1$. Recall that the package consists of circles from the same pencil and therefore all centers lie on a line. Since one center is at 0 we may assume that the line is the real axis. For five vertices, we consider two circumscribing polygons, P_1 and P_2.

  Let the five vertices in pentagon P_1 be $a = a_1 + a_2 i, b = b_1 + b_2 i, 1$ and the conjugates of a and b where $b_1 < 0$ and $0 < a_1$. Let

$$2a_1 + 2b_1 + 1 = \alpha. \tag{5}$$

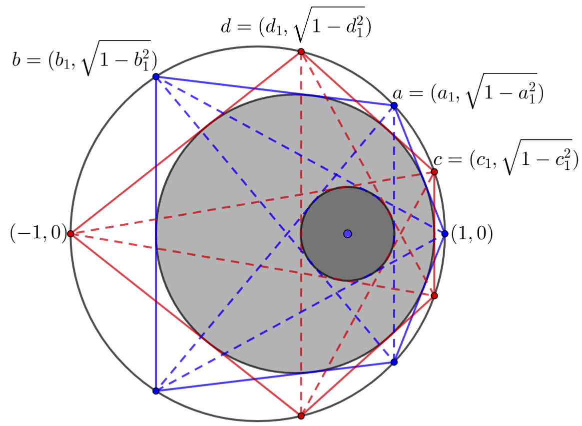

Figure 21: Illustration of the proof when $n = 4, \dim \ker A = 1$

Now let P_2 be the polygon with vertices $c = c_1 + c_2 i, d = d_1 + d_2 i, -1$ and the conjugates of c and d. Then by Theorem 27, we have

$$2c_1 + 2d_1 - 1 = \alpha. \tag{6}$$

Let us denote the circle in the package tangent to $b\bar{b}$ and $c\bar{c}$ by C_1 and C_2 the circle tangent to $d\bar{d}$ and $a\bar{a}$. By Remark 14, since all of the circles have the same pseudohyperbolic center, we can assume all the centers of the package are real and non-negative. Then the center and radius of C_1 are as follows:

$$x_1 = \frac{c_1 + b_1}{2} \tag{7}$$

$$r_1 = x_1 - b_1 = c_1 - x_1 = \frac{c_1 - b_1}{2}. \tag{8}$$

Similarly, the center and radius of C_2 are as follows:

$$x_2 = \frac{a_1 + d_1}{2} \tag{9}$$

$$r_2 = x_2 - d_1 = a_1 - x_2 = \frac{a_1 - d_1}{2}. \tag{10}$$

We then find the equation of the lines joining d and -1 and a and 1. The line joining a and 1:

$$\sqrt{1+a_1}x + \sqrt{1-a_1}y - \sqrt{1+a_1} = 0$$

and the line joining d and -1:

$$\sqrt{1-d_1}x - \sqrt{1+d_1}y + \sqrt{1-d_1} = 0.$$

Furthermore, the distance from the point $(x_1, 0)$ to the lines above is also the radius, so we must have

$$\left|\sqrt{1-d_1}\left(\frac{c_1+b_1+2}{2}\right)\right| = \left|\sqrt{1+a_1}\left(\frac{c_1+b_1-2}{2}\right)\right| = \frac{c_1-b_1}{\sqrt{2}}. \qquad (11)$$

Thus

$$\sqrt{1-d_1} = \frac{\frac{(c_1-b_1)}{\sqrt{2}}}{(1+(b_1+c_1)/2)} \qquad (12)$$

and

$$\sqrt{1+a_1} = \frac{\frac{(c_1-b_1)}{\sqrt{2}}}{(1-(b_1+c_1)/2)}. \qquad (13)$$

Having established these relations, we can now continue with the proof.

First, assume that C_1 is centered at 0. Since circles centered at 0 are rotationally symmetric about their center, the polygon circumscribing C_1 must have equally spaced vertices. Thus, C_2 is tangent to lines skipping over one vertex of a regular pentagon centered at 0. It follows that C_2 must be centered at 0 as well.

Now assume C_2 is centered at 0 so $a_1 = -d_1$. Then $\sqrt{1-d_1} = \sqrt{1+a_1}$ so setting Eq. (13) equal to Eq. (12) we have

$$\sqrt{1-d_1} = \frac{\frac{(c_1-b_1)}{\sqrt{2}}}{(1+(b_1+c_1)/2)} = \frac{\frac{(c_1-b_1)}{\sqrt{2}}}{(1-(b_1+c_1)/2} = \sqrt{1+a_1}.$$

Then

$$\frac{1}{(1+(b_1+c_1)/2)} = \frac{1}{(1-(b_1+c_1)/2}$$

so

$$1+\frac{b_1+c_1}{2} = 1-\frac{b_1+c_1}{2}$$

and we get $b_1 = -c_1$. Therefore, $x_1 = \frac{c_1+b_1}{2} = 0$ and C_1 is centered at 0.

- Assume $\dim\ker A = 2$. Since $W(A)$ is a circular disk, we may write A in the form $A = \begin{bmatrix} 0_2 & B \\ 0_2 & C \end{bmatrix}$ where $B = \begin{bmatrix} c & d \\ e & f \end{bmatrix}$ and $C = \begin{bmatrix} \alpha & b \\ 0 & \alpha \end{bmatrix}$ satisfy $B^*B + C^*C = I_2$. By Remark 29, we know there are two possible cases: the Kippenhahn curve consists of an ellipse and two points or the curve consists of two ellipses.

  We will first consider the case when the Kippenhahn curve consists of an ellipse and two points. Condition 2 of Theorem 30 gives the following relation:

  $$r^2(\lambda_i\lambda_j) = 0 \text{ where } \lambda_i, \lambda_j \text{ are the foci of the ellipse.}$$

  Then we have the circle centered at 0 (when $r^2(0) = 0$) or the circle centered at α (when we have $r^2\alpha^2 = 0$). Since r is the minor axis of the ellipse, we have $r > 0$. Thus, $\alpha = 0$ and we are done.

  Now, consider the case when the Kippenhahn curve consists of two ellipses. By Theorem 34, we know that $\sigma(C) = \{\alpha\}$. So by the Elliptical Range Theorem, $W(C)$ is circular. Now we wish to show that $W(C) \subseteq W(A)$.

Consider $v = (0, 0, \gamma, \beta)^T \in (\ker A)^\perp$ with $\|v\| = 1$. Then $Av = (\gamma c + \beta d, \gamma e + \beta f, \alpha\gamma + \beta x, \alpha\beta)^T$. So $\langle Av, v\rangle = |\gamma|^2\alpha + \overline{\gamma}\beta x + |\beta|^2\alpha$. Now consider $x = (\gamma, \beta) \in (\ker C)^\perp$ with $\|x\| = 1$. Then $Cx = (\alpha\gamma + x\beta, \alpha\beta)^T$ so $\langle Cx, x\rangle = |\gamma|^2\alpha + \overline{\gamma}\beta x + |\beta|^2\alpha = \langle Av, v\rangle$. Thus, $W(C) \subset W(A)$.

So we have four eigenvalues and we know that we have one circle and an ellipse. However, the foci of the circle must be the same. Therefore, the two foci of the ellipse must also be the same. So, the Kippenhahn curve of A consists of two circles, C_1 centered at 0 with radius r and C_2 centered at α with radius s, and we use Corollary 31. We compute and simplify the following matrix multiplications:

$$A^*A = \begin{bmatrix} 0_2 & 0_2 \\ 0_2 & I_2 \end{bmatrix},$$

$$A^*A^2 = \begin{bmatrix} 0_2 & 0_2 \\ 0_2 & \begin{matrix} \alpha & b \\ 0 & \alpha \end{matrix} \end{bmatrix},$$

$$A^*A^3 = \begin{bmatrix} 0_2 & 0_2 \\ 0_2 & \begin{matrix} \alpha^2 & 2\alpha b \\ 0 & \alpha^2 \end{matrix} \end{bmatrix}.$$

Then $tr(A) = 2a, tr(A^*A^2) = 2\alpha$, and $tr(A^*A^3) = 2\alpha^2$. Condition 1 of Corollary 31 gives $\gamma^2 = tr(A^*A) - 2\alpha^2 = 2 - 2\alpha^2$. So using condition b of Corollary 31, we have

$$\begin{aligned} r^2\alpha^2 &= (\gamma^2 + 2\alpha^2)\alpha^2 + 2\alpha^2 - (2\alpha)(2\alpha) \\ &= (2 - 2\alpha^2 + 2\alpha^2)\alpha^2 + 2\alpha^2 - 4\alpha^2 \\ &= 0. \end{aligned}$$

Thus, $\alpha = 0$ and $W(A)$ is a circular disk centered at 0. (Note that $b \neq 0$ as then A will no longer be irreducible by [GWW16, Proposition 2.6].)

- Assume $\dim\ker A = 3$. Since we know that the dimension of the kernel is 3, we can find an orthonormal basis $\{f_1, f_2, f_3\}$ for $\ker A$. Let f_4 be a unit vector orthogonal to $\{f_1, f_2, f_3\}$, so that $\{f_4\}$ is a basis for $(\ker A)^\perp$. Then $\{f_1, f_2, f_3, f_4\}$ form a basis for A and with respect to this basis we get a matrix representation

$$\begin{bmatrix} 0 & 0 & 0 & x \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & w \end{bmatrix}.$$

Now using Theorem 34, we know that two of the eigenvalues must be the same and the value will also be the center of $W(A)$. If the two eigenvalues are 0 and 0, we are done. If the two eigenvalues are 0 and w, then $w = 0$ and so $W(A)$ is centered at 0.

$\square$

4 Proof of Theorem 17

Rather than looking at all partial isometries, Spitkovsky and Wegert focused on $n \times n$ partial isometries A with $\dim \ker A = 1$. They were able to show that for all $n \geq 2$, if the numerical range of A is circular, then it must be centered at 0. They used Mirman's invariant value and elliptical integrals to prove this statement. However, we have found a simpler proof that only involves classical geometry and linear algebra.

4.1 Classical Geometry Proof

Recall $\widehat{B}(z) = zB(z)$ where $B(z) = \prod_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k}z}$ and $a_k \in \mathbb{D}$. Suppose $\widehat{B}$ has two zeros at 0 and the numerical range of S_B is circular. By Theorem 16, we know that $W(S_B)$ is inscribed in a convex polygon inscribed in $\mathbb{T}$. So, the package of circles from $W(S_B)$ are Poncelet circles. Also note that the zeros of B are the foci of the circles. Since we assumed that $\widehat{B}$ has two zeros at 0 we know that one of the zeros of B is 0. Now all of the curves in the package are circles and Mirman's result, [Mir05, Section 2], tells us that the zeros of B are the foci of the circles, so a center of one of the circles in the package is at zero. Then the following proof shows that all of the circles in the package must be centered at 0.

Proof: Let $\lfloor x \rfloor$ be the largest integer less than or equal to x. Recall that we have shown that all circles have the same pseudohyperbolic center, so we may assume that the x-coordinates of the centers are nonnegative. If C_1 is centered at 0, then the vertices of the circumscribing polygon are equally spaced and all circles in the package will be centered at 0. Now suppose for some $k = 2, 3, \ldots, \lfloor \frac{n}{2} \rfloor$, C_k is centered at 0.

First, if C_k is a point, then consider the vertical line tangent to C_1 passing through the points $a = (a_1, \sqrt{1-a_1^2})$, and $\overline{a}$ where $a_1 \geq 0$. Since $C_k = \{0\}$, the point identified through C_k from a must be $-a$ and the point identified via C_k (through 0) from $\overline{a}$ must be $-\overline{a}$. Then the center of C_1 is $\frac{a+(-a)}{2} = 0$ and we are done.

For C_k with radius $r_k > 0$, we choose a polygon P^* circumscribing C_k that is inscribed in $\mathbb{T}$ (not necessarily convex) with $f_1 = (1, 0)$ as one of the vertices. Let $f = \{f_1, f_2, \dots, f_l\}$ be the set of vertices of P^*. Note that the vertices of every polygon circumscribing C_k are equally spaced. Call the second vertex in the polygon P^* for C_k: $f_2 = e^{i\alpha} = (\cos\alpha, \sin\alpha)$ where $0 < \alpha < \pi$. We search for the next vertex in a circumscribing polygon P_1 for C_1 with $(1, 0)$ as one of the vertices, where C_1 is the largest circle in the package (that is inscribed in a convex polygon inscribed in $\mathbb{T}$). Call the successor vertex $g_1 = e^{i\theta} = (\cos\theta, \sin\theta)$. Since C_1 is symmetric with respect to the real line and $n \geq 3$, we have $0 < \theta < \pi$. Let $\widehat{P}$ be the circumscribing polygon to C_k with g_1 as one of its vertices. Let $g = \{g_1, g_2, \dots, g_l\}$ be the set of vertices of $\widehat{P}$. Let the vertex after g_1 in the polygon $\widehat{P}$ for C_k: $g_2 = e^{i(\theta+\alpha)} = (\cos(\theta+\alpha), \sin(\theta+\alpha))$ where $0 < \alpha < \pi$ (see figure 22). Note that α is the same for the vertices in f and vertices in g because C_k is centered at 0.

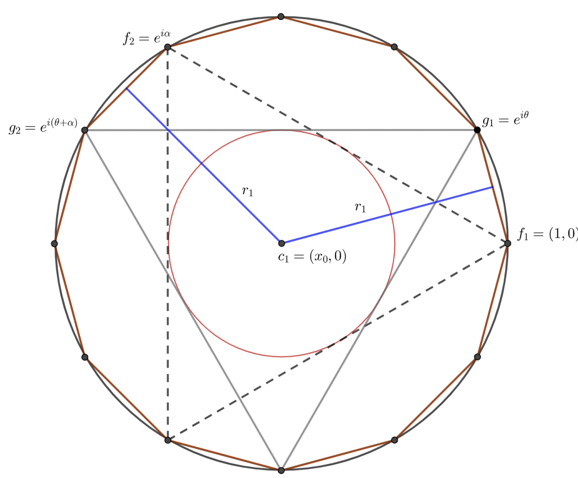

Figure 22: Illustration of the geometric proof of Theorem 17

So the line segment f_1g_1 is a rotation of f_2g_2. Now C_1 is tangent to both f_1g_1 and f_2g_2 as f_j and g_j are neighboring points in the corresponding polygon for C_1. Let us denote the center of C_1 by $c_1 = (x_0, 0)$.

Consider the line segment f_1g_1. The equation of that line is

$$(\cos\theta - 1)y - (\sin\theta)x + \sin\theta = 0.$$

Now consider the line segment f_2g_2. Then the equation of the line is

$$\begin{aligned}(\cos(\theta+\alpha) - \cos\alpha)y - (\sin(\theta+\alpha) - \sin\alpha)x + \\ (-\sin\alpha(\cos(\theta+\alpha) - \cos\alpha) + (\sin(\theta+\alpha) - \sin\alpha)\cos\alpha) = 0. \quad (14)\end{aligned}$$

By the addition formulas for sine and cosine, the constant in (14) becomes $\sin\theta$ so (14) simplifies to

$$(\cos(\theta+\alpha) - \cos\alpha)y - (\sin(\theta+\alpha) - \sin\alpha)x + \sin\theta = 0. \quad (15)$$

The distance from $c_1 = (x_0, 0)$ to both f_1g_1 and f_2g_2 is the radius of C_1. Thus,

$$r_1 = \frac{|-x_0\sin\theta + \sin\theta|}{\sqrt{2(1-\cos\theta)}} \quad (16)$$

and

$$r_1 = \frac{|-(\sin(\theta+\alpha) - \sin\alpha)x_0 + \sin\theta|}{\sqrt{(\cos(\theta+\alpha) - \cos\alpha)^2 + (\sin(\theta+\alpha) - \sin\alpha)^2}}. \quad (17)$$

By the sine addition formula, (17) simplifies to

$$r_1 = \frac{|(\sin\alpha - \sin\theta\cos\alpha - \cos\theta\sin\alpha)x_0 + \sin\theta|}{\sqrt{2(1-\cos\theta)}}. \quad (18)$$

We wish to show x_0 must be 0 for both (16) and (17) to hold. To this end, note that $-x_0 \sin\theta + \sin\theta > 0$. We now show

$$(\sin\alpha - \sin\theta\cos\alpha - \cos\theta\sin\alpha)x_0 + \sin\theta > 0$$

for $0 < \theta, \alpha < \pi$. The inequality above becomes

$$\sin\alpha(1 - \cos\theta)x_0 + \sin\theta(1 - (\cos\alpha)x_0) > 0.$$

Since $0 < x_0 < 1$ and $0 < \theta, \alpha < \pi$, we have $\sin\alpha(1 - \cos\theta)x_0 > 0$ and $\sin\theta(1 - x_0\cos\alpha) > 0$. Thus, $(\sin\alpha - \sin\theta\cos\alpha - \cos\theta\sin\alpha)x_0 + \sin\theta > 0$ indeed.

Now setting (16) and (17) equal to each other, we get

$$(\sin\alpha - \sin\theta\cos\alpha - \cos\theta\sin\alpha)x_0 = -(\sin\theta)x_0. \tag{19}$$

If $x_0 \neq 0$, then

$$\sin\alpha - \sin\theta\cos\alpha - \cos\theta\sin\alpha = -\sin\theta$$

so

$$\sin\alpha(1 - \cos\theta) + \sin\theta(1 - \cos\alpha) = 0. \tag{20}$$

But we showed above that $\sin\alpha(1-\cos\theta) > 0$ and similarly, we have $\sin\theta(1-\cos\alpha) > 0$ so $\sin\alpha(1-\cos\theta)+\sin\theta(1-\cos\alpha) > 0$. Thus, $x_0 = 0$ so C_1 is centered at the origin. It follows that the vertices of the polygon for C_1 is equally spaced and as above, C_j where $j = 2, 3, \ldots, \lfloor \frac{n}{2} \rfloor$ is centered at 0 as well. Since the foci of the curves (i.e., the centers of the circles) are the zeros of the Blaschke product, all zeros must be zero.

Remark 35 *In the case when the polygon circumscribing C_k has $n + 1$ vertices on*

$\mathbb{T}$, $f_2 = g_1$ in the proof above and we are actually discussing a chord tangent to C_1.

□

4.2 Projective Geometry Proof

Proposition 36 *Given a Poncelet package of circles, if one of the circles is centered at* 0*, then all the circles in the package are centered at* 0.

Proof: Suppose one of the circles is centered at $(c, 0)$, where $c > 0$. Then we know that I_C is invariant of the choice of circle, so

$$1 + I_C = \frac{2c + 1 + c^2 - R^2}{2c} = \frac{(1+c)^2 - R^2}{2c}$$

will also be invariant. Hence if one circle is not centered at zero, the others cannot be. Thus, all other circles must also be centered at 0.

□

5 Conclusion

In 2016, Gau, Wang, and Wu asked the following question: "which circular disks contained in $\overline{\mathbb{D}}$ are the numerical range of a partial isometry on a finite dimensional space [GWW16]?" This question became the cornerstone of two main results by Gau, Wang, and Wu, and Spitkovsky and Wegert. A result by Tabachnikov and Schwartz, which we also provide a simple proof of, provides insight into the geometry of these problems. In this thesis, we aimed to re-prove and clarify these three results.

We first re-proved the following result by Gau, Wang, and Wu.

(Theorem 8) *Let A be an $n \times n$ partial isometry and $n \leq 4$. If $W(A) = \{z \in \mathbb{C} : |z - c| \leq r\}$ where $r > 0$, then $c = 0$.*

Their proof goes through each dimension $n = 2, 3, 4$ and for each n, they prove this result for all possible dimensions of the kernel of the operator. In addition, their proofs needed clarification. In Section 3, we provide a new proof and, we hope that we also clarify their proofs. Our work has significantly simplified the extreme cases – when the dimension of the kernel is 1 and $n - 1$.

In 2021, Spitkovsky and Wegert [WS21] proved Gau, Wang, and Wu's statement for all n for a certain class of partial isometries whose dimension of the kernel is 1 (Theorem 17). Their proof uses elliptic integrals, which is a natural technique given the history of the problem. However, it is also more complicated than the geometric proofs that we provide. By relating these partial isometries to *compressions of the shift* operators, we were able to provide a simpler and equivalent statement.

(Theorem 17) *Suppose $\mathbb{T}$ and circle C_1 inside $\mathbb{T}$ form a Poncelet package of circles. If one of the circles in the Poncelet package is centered at 0, then all the circles are centered at 0.*

By doing so, we turn their elliptic integral approach into a purely linear algebra and geometric problem (see Section 4.1). Further, we provide an explanation for a statement made by Mirman [Mir12] about an invariant of the circles in a Poncelet package,

$$I_C = \frac{1 + c^2 - R^2}{2c},$$

where c denotes the center of the Poncelet circle C_1 and R denotes the radius.

Lastly, in Section 3.1, we also give a new proof of a result due to Tabachnikov and Schwartz [ST16] that shows that the sum of the vertices of the circumscribing polygons is constant.

In future work, we hope to extend Theorem 8 to $n \geq 6$. The case when $n = 5$ was covered by Benharrat and Mehdi in 2023 [BM23] using techniques similar to that of Gau, Wang, and Wu. We also hope to clarify results by Mirman that finds a formula for the centers of the circles in a Poncelet package of circles. Unfortunately, due to the time constraints, we were unable to explores these topics further. We may continue to work in this direction in the future.

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