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Positivity Among P-Partition Generating Functions of Partially Ordered Sets

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POSITIVITY AMONG P-PARTITION GENERATING FUNCTIONS OF PARTIALLY ORDERED SETS

by

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Abstract

We find necessary and separate sufficient conditions for the difference between two labeled partially ordered set's (poset) partition generating functions to be positive in the fundamental basis. We define the notion of a jump sequence for a poset and show how different conditions on the jump sequences of two posets are necessary for those posets to have an order relation in the fundamental basis. Our sufficient conditions are of two types. First, we show how manipulating a poset's Hasse diagram produces a poset that is greater according to the fundamental basis. Secondly, we also provide tools to explain posets that are constructed by combining other posets in certain ways through the so-called Ur-operation. Finally, we are able to provide both necessary and sufficient conditions for positivity among posets of Greene shape $(k, 1)$ and among a subclass of caterpillar posets, and a complete (and graphically pleasing) representation of the order relations between posets of the former type.

Chapter 1

Introduction

Perhaps the most widely known open question in Combinatorics is “What is Combinatorics?” It is an incredibly varied field of mathematics that has the tendency to arise other fields. One proposed answer to this question is “The study of finite sets.” Two important tools of study in Cominatorics are generating functions and ordered sets. The focus of this thesis is primarily in the combination of these two, generating functions *for* ordered sets.

Most are familiar with the concept of total orders. These are sets in which every single pair of elements has an order relation; two elements are either equal to each other or one is larger. The integers ($\mathbb{Z}$), rational numbers ($\mathbb{Q}$), and the real numbers ($\mathbb{R}$) all have this familiar total order on them. We are focusing on the generating functions for *partially* ordered sets. These sets have order relations on them, but not *every* element is related to every other. Some elements are less than others, some are greater, but some pairs of elements have no relation to each other at all. Detailed examples and explanations of these objects can be found in Chapter 2.

Combinatorists are often concerned with counting how many elements are in whatever finite set they happen to be studying. One of the most powerful tools for doing this is generating functions. Say that you were trying to count an object determined by some parameter n, and the size of this object was $f(n)$. The generating function for this object would be $\sum_{n\geq 0} f(n)x^n$. This representation allows combinatorists to use algebraic techniques to solve problems rather than strictly combinatorial ones. For example for the Fibonacci numbers, which are defined as $a_0 = 0$, $a_1 = 1$, and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$, it is natural to ask what is the nth number in this sequence. One could iteratively calculate every element up to n, but what if n was 100? How about $1,000,000$? These calculations can quickly get out of hand, but using algebraic techniques afforded by generating functions, one can get an explicit formula for the nth Fibonacci number. This explicit formula allows us to compute the nth Fibonacci number without computing *every* element that comes before it.

Two important objects of study in Combinatorics are partitions and compositions of an integer n. Partitions of n are sequences of positive integers that sum to n where order doesn't matter. For example $(3, 2, 5)$ and $(5, 3, 2)$ are the same partition of 10. It is standard to write them in decreasing order. Compositions of n on the other hand are sequences of positive integers that sum to n where order does matter. In the previous example $(3, 2, 5)$ and $(5, 3, 2)$ are the same partition of 10 (although we wouldn't write it in the first way), but they are different compositions of 10. Our main object of study are P-partitions introduced by Stanley [Sta71], which lie somewhere in between partitions and compositions. They use a partially ordered set (which is what the P references) to partially order sequences of integers that sum to n. The main goal of this thesis is to find necessary and/or sufficient conditions for the P-partition generating functions for two partially ordered sets to be related to each other in a certain fashion (that we will explain in the following section).

Many of our results were motivated by calculations performed with [The18]. This is an open-source mathematical software that allowed us to generate all posets of a certain size, then determine whether or not one's P-partition generating functions were related to one another. This was possible up to posets of size 6, beyond which calculations took or would take several months or more to complete. Even in the case of comparing two posets alone this problem becomes intractible rather quickly.

Our research is motivated by a special case that has received a lot of attention. We will not include the specific details of this special case as it is well beyond the scope of this thesis, but this special case is motivated itself by symmetric functions. The symmetric functions have several bases, but the Schur basis is arguably the most important as Schur functions arise naturally in several areas of mathematics. It is important then to understand what occurs when two Schur functions are multiplied. The famed Littlewood-Richardson rule shows that the product of two schur basis elements s_λ and s_μ has positive coefficients when expanded in terms of Schur functions. Thus $s_\lambda s_\mu$ is called *Schur-positive*. The next natural question to arise from this asks when the differences between two such products are also Schur-positive. This question is the subject of [BBR06], [Kir04], [LPP07],[Oko97], [RS06], [LP07], and [DP07]. A natural step beyond products of Schur functions is to look at skew Schur functions and ask the analogous question: when is the difference positive? This is the focus of [KWvW08], [McN08], [McN14], [MvW09], and [MvW12].

Our generating function $K_{(P,\omega)}$ is a partially ordered set analogue of skew schur functions (and hence products of Schur functions), and includes both as (very) special cases. As posets are fundamental in combinatorics, $K_{(P,\omega)}$ is worthy of study in its own right. Many of our specific questions however are inspired by ideas from the Schur function case and the progress we make fits appropriately within this classical framework. Our work also builds on several papers which have looked at $K_{(P,\omega}$ and also asked when two generating functions for different posets are equal, as in [Fér15], [MW14], [IW18], [IW19], and [BHK17]. Closest in study to this thesis is [LP08], which investigates positivity in the more restricted case of cell transfer between disjoint partially ordered sets.

The thesis is structured as follows. In Chapter 2 we provide the necessary mathematical background on partially ordered sets, labeled posets, P-partitions, and their generating functions. Chapter 3 details some of the strictly necessary conditions that we proved for our poset relation to hold. These are conditions that *must* be true for two posets' generating functions to be related, but the mere fact that they are satisfied does not mean we actually have that relation. Chapter 4 contains a collection of related sufficient conditions and proofs that completely explain certain classes of poset relations. Note that these conditions will not be necessary for *all* poset relations, but showing that they are true will guarantee that the relation holds. Chapter 5 provides proofs of certain tools one can use to combine together posets whose relations are already known in order to get larger posets where that relation also holds. Chapter 6 details and proves the validity of a visual process that at times can simplify the process of telling whether or not two posets are related. Finally, Chapter 7 gives some examples of classes of posets wherein our necessary conditions are also strong enough to be sufficient.

Chapter 2

Mathematical Background

In this section we give the necessary background information about labeled posets and their generating functions.

2.1 Partially Ordered Sets

Given a set, it is normal to associate some sort of order on its elements. For example given the set $\{1, 2, 3, 4\}$, the usual ordering is $1 < 2 < 3 < 4$. This is an example of a totally ordered set. An important visualization of ordered sets are Hasse diagrams, where relationships between elements are denoted by lines between nodes and the direction of those relations are given by height. For example our set $\{1, 2, 3, 4\}$ with the usual ordering would have the Hasse diagram shown in Figure 2.1.

There is no need to order elements of a set so simply however. Take the set $\{1, 2, 3, 5, 6, 10, 15, 30\}$ of divisors of 30 ordered by divisibility (Figure 2.2 (a). For example, 3 is "less than" 30 because 3 divides 30. Even though $3 \leq 5$ in the usual ordering, 3 is not "less than" 5 here because 3 does not divide 5. This is why we

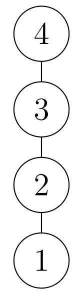


Figure 2.1: The Hasse diagram of $\{1, 2, 3, 4\}$ under the usual order

call such structures partially ordered sets (posets): not every pair of elements has a relation between them.

Definition 2.1.1. A *partially ordered set* (poset) is a set P and an order relation $\leq_P$ where the following conditions hold:

1. $a \leq_P a$ for all $a \in P$ (Reflexivity),
2. If $a \leq_P b$ and $b \leq_P a$ then $a = b$ (Antisymmetry),
3. If $a \leq_P b$ and $b \leq_P c$ then $a \leq_P c$ (Transitivity).

A poset is called *totally ordered* if in addition to the previous three conditions the following holds:

4. For any $a, b \in A$, either $a \leq_P b$ or $b \leq_P a$ (Comparability).

We can look at another poset: the subsets of $\{a, b, c\}$. The order we impose on this one is that one set A is less than another B if it is contained within it (i.e. $A \leq B$ if and only if $A \subseteq B$). This poset can be seen in Figure 2.2 (b). We can see that the structure of the poset in Figure 2.2(a) is actually the same as that in Figure 2.2(b), despite each poset containing different elements. Thus they can be treated as the same in many respects. Because we wish our results to be as general

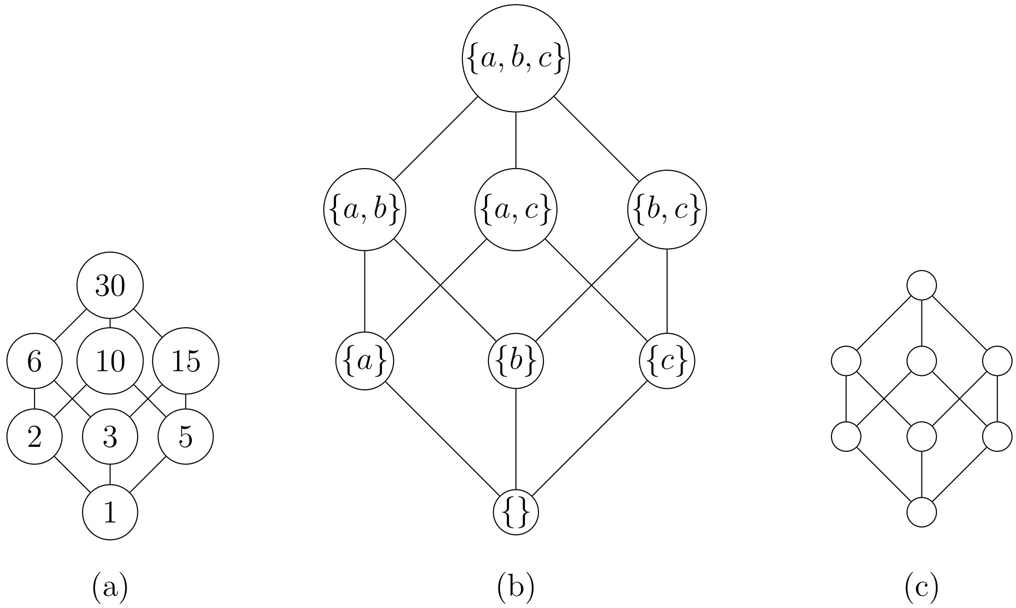


Figure 2.2: (a) The Hasse diagram of $\{1, 2, 3, 5, 6, 10, 15, 30\}$ ordered by divisibility.
(b) The Hasse diagram of the subsets of $\{a, b, c\}$ ordered by containment.
(c) The abstract poset representing both of these structures.

as possible, we consider posets as abstract objects having "elements" and "edges" instead of specifying what those elements might represent. So instead of restricting our attention to the specific posets shown in Figure 2.2(a) and 2.2(b), we can study the generic poset in Figure 2.2(c).

The cardinality (size) of a poset will be denoted $|P|$. For example all three posets in Figure 2.2 have cardinality 8. The order relation for a given poset P will be denoted $\leq_P$. For example, for the poset P in Figure 2.2 (a), we would write that $3 \leq_P 6$ and $3 \nleq_P 10$. At times we may omit the subscript P if it is clear from context.

2.2 (P, ω)-partitions

On a poset P, we can attach numbers to the elements of P to form what is called a P-partition. More formally, a P-partition is a map f from the elements of a poset P to the positive integers that is order-preserving. That is to say if a and b are elements of P where $a \leq_P b$, then $f(a) \leq f(b)$. For example Figure 2.3 shows several valid P-partitions of a poset.

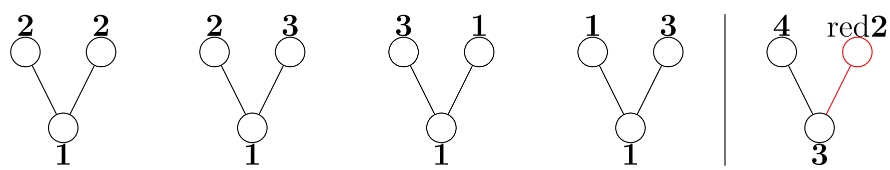


Figure 2.3: A poset P with four (of many possible) different P-partitions, and one assignment of numbers that breaks the rules for P-partitions.

Here however we want more structure on our P-partition. Specifically, we designate some of the edges as "strict," denoted by double lines in Figure 2.4 (a). If $a < b$ is a strict edge, we require $f(a)$ to be strictly less than $f(b)$, i.e. $f(a) < f(b)$.

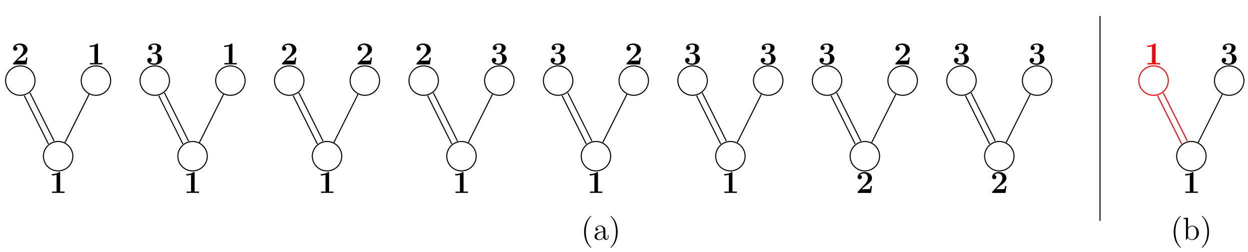


Figure 2.4: (a) A poset P with weak and strict edges and all of the P-partitions that can be made with $\{1, 2, 3\}$.
(b) That same poset with a P-partition that worked for the unlabeled version, but not with our new labels.

Figure 2.4 (a) is an example of the same poset with all of the different P-partitions that can be made using only $\{1, 2, 3\}$. In terms of the diagram, single lines are allowed to have the same number above and below them, where as double lines are not. The single lines we will designate as "weak" edges.

When we designate strict and weak edges, we do so by imposing an underlying labeling on a poset. A labeling ω on a poset is an injective function from the elements of a poset to a totally ordered set (normally $\{1, 2, ..., |P|\}$ under the usual ordering). From that labeling, we designate strict edges as those edges where the element larger in the poset has the lesser label, and weak edges as edges where the element larger in the poset has the larger label. For example, the poset in Figure 2.4 (a) can be labeled with the totally ordered set $\{1, 2, 3\}$ under the usual order to get the labeled poset in Figure 2.5.

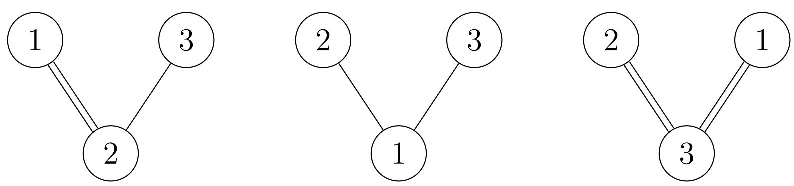


Figure 2.5: Three different ways (of several) to label a particular poset and get weak and strict edges.

When given an abstract poset P and a labeling ω, we denote the combined labeled poset as (P, ω). Now that we have this labeling, instead of talking abstractly about "weak" and "strict" edges we can define our P-partitions in terms of the labeling function we have on our poset.

Definition 2.2.1. For a labeled poset (P, ω), a (P, ω)-*partition* is a map f from P to the positive integers satisfying the following two conditions:

(i) if $a \leq_P b$, then $f(a) \leq f(b)$;

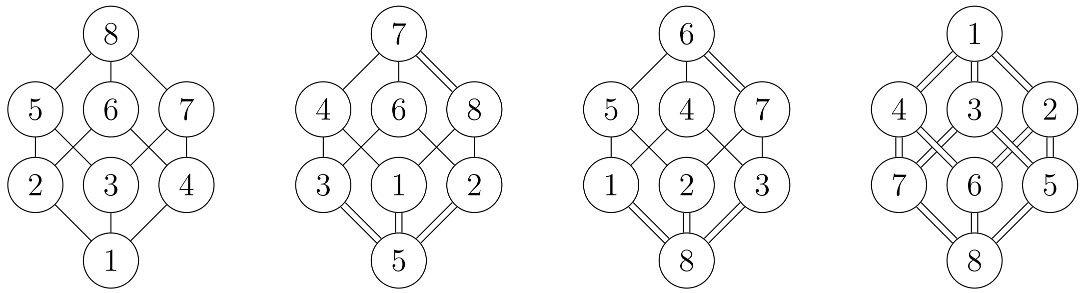


Figure 2.6: The same poset with several different labeling functions, and the different strict and weak edges that result from them.

(ii) if $a \leq_P b$ and $\omega(a) > \omega(b)$, then $f(a) < f(b)$.

For $a, b \in P$, we say that a is covered by b if $a <_P b$ and there is no $c \in P$ such that $a <_P c <_P b$. We refer to the relation between a and b as an *edge* because these covering relations are exactly the edges of the Hasse diagram. Note that every relation between elements of a poset is capable of being represented by a series of cover relations. Thus the cover relations alone are a sufficient representation of an entire poset. Additionally, it will sometimes be easier to notate for us to reference elements of the labeled posets by their labels. We will always note beforehand when we are going to use this style of reference.

Because of the requirements on a (P, ω)-partition in Definition 2.2.1, we say that an edge $a <_P b$ of a poset is *weak* if $\omega(a) < \omega(b)$, and *strict* if $\omega(a) > \omega(b)$. This is the origin of our weak and strict edges from the Hasse diagrams before. For examples see Figures 2.5 and 2.6.

As we can see from Figure 2.6, the same poset P with different labelings can result in different strict and weak relations. Similarly, different labelings of a poset can (and often do) result in the exact same collection of weak and strict edges. This may seem problematic, but the flexibility afforded in the labels is vital for several of our results. As we will often wish to work from an abstract labeling rather than a fixed one, we require terminology for our different representations:

Definition 2.2.2. An labeling τ of a labeled poset (P, ω) is *valid* alternative labeling if it respects all edge relations, meaning that for all elements $a, b \in P$ where b covers a

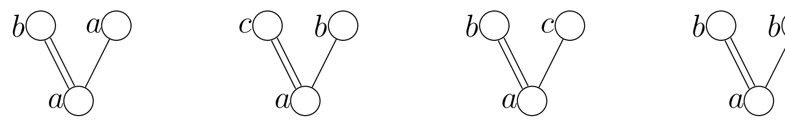
- if $\omega(a) < \omega(b)$ then $\tau(a) < \tau(b)$, and
- if $\omega(a) > \omega(b)$ then $\tau(a) > \tau(b)$.

Since the labelings in this sense only determine which edges are weak and which are strict, and are not unique, most of our diagrams will omit the labelings in favor of only showing the weak and strict edges. We refer to a labeled poset as being *naturally labeled* if its labeling implies all cover relations are weak.

2.3 Generating Functions

Given a labeled poset such as the one in Figure 2.4, there are an infinite number of valid (P, ω) partitions. This is because the numbers that we attach to these elements can be any positive integer; they do not only have to come from the set $\{1, 2, 3\}$ as was the case in Figure 2.4 (a). *Any* trio of positive integers that respects those weak and strict relations is a valid (P, ω)-partition. It is natural to desire a more tangible representation of all of these partitions.

We can achieve this through *generating functions.* Transferring an unwieldy mathematical construct to a generating function is a good way to get a more concrete representation of that object. It also allows us to use algebraic techniques on those functions in order to make claims that otherwise may be very difficult to state or prove. Let us first look again at the poset in Figure 2.4. One can see that every (P, ω)-partition of the poset is one of the four following types, where a, b, and c are positive integers:

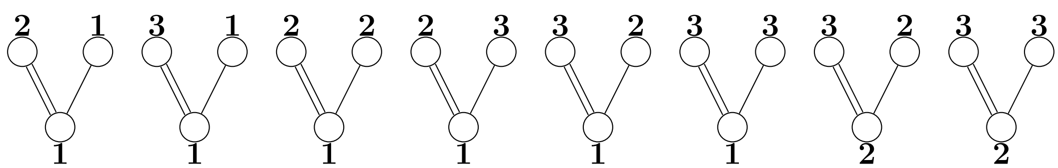


$$a < b \qquad a < b < c \qquad a < b < c \qquad a < b$$

$$x_a^2 x_b \qquad x_a x_b x_c \qquad x_a x_b x_c \qquad x_a x_b^2 \tag{2.1}$$

The inequalities that are written below each of the posets are the restrictions placed upon a, b, and c. To verify these are correct, simply take a, b, and c to be integers from $\{1, 2, 3\}$, and fill in the posets following the rules written below them. You will recover all of the (P, ω)-partitions that are shown in Figure 2.4 (a).

The monomials that are written below the rules are the components of the generating function that we are attempting to create. The power on each x_i is the frequency at which i appears in that (P, ω)-partition. Concretely, if we were to do this with the (P, ω)-partitions from Figure 2.4 (a), then each (P, ω)-partition would have the following algebraic representation:



$$x_1^2 x_2 \quad x_1^2 x_3 \quad x_1 x_2^2 \quad x_1 x_2 x_3 \quad x_1 x_2 x_3 \quad x_1 x_3^2 \quad x_2^2 x_3 \quad x_2 x_3^2$$

One can check that each of these monomials falls in to one of the four types mentioned before.

Before our formal definition of the (P, ω)-partition generating function, note that for a (P, ω)-partition f, the size of the inverse-image of a number a, denoted $|f^{-1}(a)|$, is the number of elements of P that f maps to a

Definition 2.3.1. For a labeled poset (P, ω), we define the (P, ω)-*partition generating* *function* $K_{(P,\omega)} = K_{(P,\omega)}(x_1, x_2, ...)$ by

$$K_{(P,\omega)} = \sum_{(P,\omega)\text{-partition } f} x_1^{|f^{-1}(1)|} x_2^{|f^{-1}(2)|} \cdots$$

where the sum is over all (P,ω)-partitions.

Now that we have the formal definition of the (P,ω)-partition generating function, we construct the (P,ω)-partition generating function for the poset from (2.1). We know that every one of the (P,ω)-partitions takes one of the four forms we identified here. The sum for each of these individual forms is given by the rules and algebraic representation below the Hasse diagrams. Thus we have that

$$K_{(P,\omega)}(x) = \sum_{a<b} x_a^2 x_b + \sum_{a<b<c} x_a x_b x_c + \sum_{a<b<c} x_a x_b x_c + \sum_{a<b} x_a x_b^2 = \sum_{\substack{i \leq j \leq k \\ i<k}} x_i x_j x_k. \quad (2.2)$$

These summations give an algebraic representation for every (P,ω)-partition, and we can write it neatly as the latter sum. The following section provides more information on how these functions behave, and how they are represented in this thesis.

2.4 Quasisymmetric functions

All (P,ω)-partition generating functions are types of functions called quasisymmetric functions. Quasisymmetric functions have a rich structure, and we will manipulate and compare the (P,ω)-partition generating functions of different posets by means of these quasisymmetric-function representations.

Definition 2.4.1. A *quasisymmetric function* in the variables $x_1, x_2, \ldots$ with rational coefficients, is a formal power series $g = g(x) \in \mathbb{Q}[[x_1, x_2, \ldots]]$ of bounded degree such that for every sequence $a_1, a_1, \ldots a_k$ of positive integer exponents, we have that the (nonzero) coefficient of $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}$ is equal to the (nonzero) coefficient of $x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_k}^{a_k}$ whenever $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$.

For an example, take the polynomial

$$2x_1^2x_2x_3^3 + 2x_1^2x_2x_4^3 + 2x_1^2x_3x_4^3 + 2x_2^2x_3x_4^3.$$

The sequence of positive integer exponents is $2, 1, 3$. We are working with $x_1,\ x_2,\ x_3$, and x_4. The ways to choose three of these variables so that the indices are strictly increasing are $x_1x_2x_3$, $x_1x_2x_4$, $x_1x_3x_4$, and $x_2x_3x_4$. The coefficients of each of $x_{i_1}^2x_{i_2}^1x_{i_k}^3$ when $i_1 < i_2 < i_3$ are the same, which is what makes a polynomial quasisymmetric.

Now look at the generating functions for labeled posets. For any specific (P, ω)-partition, one can replace the numbers $i_1 < i_2 < \cdots < i_k$ to which the partition maps with any strictly increasing series of numbers. See the poset in Figure 2.7. Note that

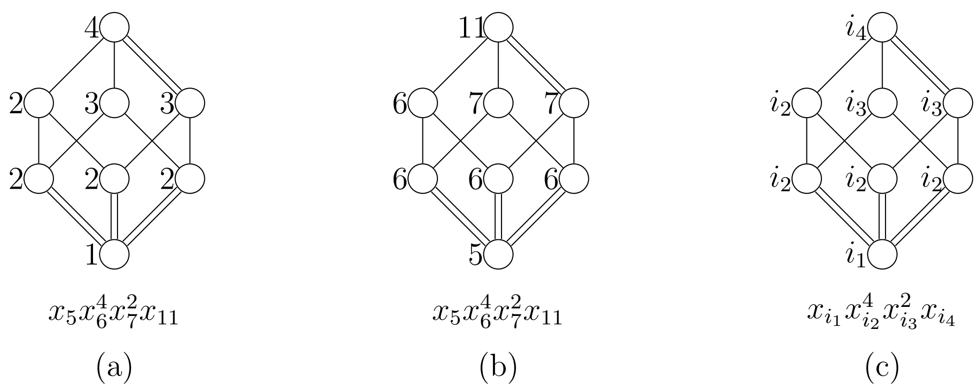


Figure 2.7: For a poset (P, ω):

(a) A (P, ω)-partition of the poset.

(b) Another partition with a different series of elements

(c) A general way to construct this type of partition, with $i_1 < i_2 < i_3 < i_4$.

for (c), the nature of the general way to construct the partition is that any sequence of positive $i_1 < i_2 < i_3 < i_4$ can be chosen, and the resulting contribution to the (P, ω)-partition generating function is therefore quasisymmetric.

We will make use of two bases for quasisymmetric functions in this thesis. The first is the *monomial quasisymmetric function* M_α given by

$$M_\alpha = \sum_{i_1<i_2\cdots<i_k} x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k},$$

where $\alpha = (a_1, a_2, ..., a_k)$. We will call the way of writing a quasisymmetric function in terms of its M-basis elements that function's *M-expansion.* For example, $M_{32} = \sum_{i_1<i_2} x_{i_1}^3 x_{i_2}^2$.

The second basis requires several introductory definitions to properly define.

Definition 2.4.2. A *composition* of a series of n is a series of positive integers $a_1, a_2, ..., a_k$ such that $\sum_{i=1}^{k} a_i = n$.

Definition 2.4.3. A *refinement* of a composition $(a_1, a_2, ..., a_k)$ is a sequence of positive integers $(b_1, b_2, ..., b_l)$ such that

$$(a_1, a_2, ..., a_k) = \left(\sum_{i=1}^{\ell_1} b_i, \sum_{i=\ell_1}^{\ell_2} b_i, ..., \sum_{i=\ell_{k-1}}^{\ell_k} b_i \right).$$

For example, one can think of a refinement of a composition as a "breakdown" of a composition. If one were to take the composition $(3, 3, 3)$ of 9, then $(2, 1, 3, 3)$, $(3, 1, 2, 3)$, $(3, 3, 1, 1, 1)$ would all be refinements. The composition $(4, 2, 3)$ on the other hand would not be a refinement. The sequence of n ones $(1, 1, ..., 1)$ is a refinement of any composition of n.

The second basis is the *fundamental quasisymmetric functions* F_α, given by

$$F_\alpha = \sum_{\beta} M_\beta, \tag{2.3}$$

where β is any refinement of α. For example, $F_{3,1} = M_{3,1} + M_{2,1,1} + M_{1,2,1} + M_{1,1,1,1}$. We will call the way of writing a quasisymmetric function in terms of its F-basis elements that function's *F-expansion.*

The M-expansion of the (P, ω)-partition generating function can be computed by noting all of the different ways that a (P, ω)-partition can be assigned. We have actually already computed the M-expansion of the poset (P, ω) represented in (2.1). Constructing the different forms that a (P, ω)-partition can take is how you compute the M-basis expansion. Thus we have that in this case,

$$K_{(P,\omega)} = \sum_{a<b} x_a^2 x_b + \sum_{a<b<c} x_a x_b x_c + \sum_{a<b<c} x_a x_b x_c + \sum_{a<b} x_a x_b^2 = M_{2,1} + 2M_{1,1,1} + M_{1,2}.$$

By inspection of the M-expansion, or more easily using Theorem 2.4.9 below, we determine that the F-expansion is

$$K_{(P,\omega)} = F_{2,1} + F_{1,2}.$$

Definition 2.4.4. We say that a quasisymmetric function g is *F-positive*, denoted $g \geq_F 0$, if all of the coefficients of its F-basis expansion are non-negative.

As a consequence of Theorem 2.4.9 below, all (P, ω)-partition generating functions are F-positive.

It is the goal of this thesis to determine conditions for F-positivity in the difference between two (P, ω)-partition generating functions. In the notation, we wish to study when labeled posets (P, ω) and (Q, τ) satisfy $K_{(Q,\tau)} - K_{(P,\omega)} \geq_F 0$, also denoted $K_{(Q,\tau)} \geq_F K_{(P,\omega)}$.

Definition 2.4.5. For labeled posets (P, ω) and (Q, τ), we say that P is *F-less than or equal to Q*, denoted $P \leq_F Q$, if $K_{(Q,\tau)} - K_{(P,\omega)} \geq_F 0$. Define *$M$-less than or equal to* similarly, and denote it by $P \leq_M Q$.

From this point on we will only consider relations $P \leq_F Q$ or $P \leq_M Q$ in the case when $|P| = |Q|$, since otherwise P and Q are incomparable.

Definition 2.4.6. The *F-support* of (P, ω), denoted $\mathrm{supp}_F(P, \omega)$, is the set of compositions α such that F_α appears with nonzero coefficient in the F-expansion of $K_{(P,\omega)}$. Define the *M-support* analogously.

For example the generating function that we found for (2.1) implies that that poset has an F-support of $\{(2, 1), (1, 2)\}$ and an M-support of $\{(2, 1), (1, 2), (1, 1, 1)\}$. Comparing the F-support of posets is often much easier to handle than the $\leq_F$ relation, because it is generally easier studying containment rather than positivity. The following proposition connects many of the different ways of comparing the generating functions of posets.

Proposition 2.4.7. *Let (P, ω) and (Q, τ) be labeled posets. Then*

- *If $(P, \omega) \leq_F (Q, \tau)$ then $(P, \omega) \leq_M (Q, \tau)$.*
- *If $(P, \omega) \leq_F (Q, \tau)$ then $\mathrm{supp}_F(P, \omega) \subseteq \mathrm{supp}_F(Q, \tau)$.*
- *If $(P, \omega) \leq_M (Q, \tau)$ then $\mathrm{supp}_M(P, \omega) \subseteq \mathrm{supp}_M(Q, \tau)$.*
- *If $\mathrm{supp}_F(P, \omega) \subseteq \mathrm{supp}_F(Q, \tau)$ then $\mathrm{supp}_M(P, \omega) \subseteq \mathrm{supp}_M(Q, \tau)$*

How we get the F-support from the F-basis and how we construct the F-basis from the M-basis directly implies these relations. None of these implications hold in the converse direction.

This proposition combined with Theorem 2.4.9 gives us a collection of implications that is represented in the following diagram:

$$\begin{array}{ccc}
\mathcal{L}(P,\omega) \subseteq \mathcal{L}(Q,\tau) & & \\
\Downarrow & & \\
(P,\omega) \leq_F (Q,\tau) & \Rightarrow & \mathrm{supp}_F(P,\omega) \subseteq \mathrm{supp}_F(Q,\tau) \\
\Downarrow & & \Downarrow \\
(P,\omega) \leq_M (Q,\tau) & \Rightarrow & \mathrm{supp}_M(P,\omega) \subseteq \mathrm{supp}_M(Q,\tau)
\end{array}$$

This observation will drive much of our necessary conditions for inequality. As it turns out, examining F-positivity is a related problem of examining simpler combinatorial objects which we introduce next. The definition is rather intricate, see the example that follows for the essence.

Definition 2.4.8. A *linear extension* of a labeled poset (P,ω) is a permutation $(a_1, a_2, ..., a_{|P|})$ of $(1, 2, ..., |P|)$ such that for any elements a_i and a_j, if $\omega^{-1}(a_i) \leq_P \omega^{-1}(a_j)$ then $i < j$. We denote the set of all linear extensions of (P,ω) as $\mathcal{L}(P,\omega)$.

For example the labeled poset given in Figure 2.8 has five linear extensions: $(1,2,3,4), (2,1,3,4), (1,2,4,3), (2,1,4,3)$, and $(2,4,1,3)$. Roughly, a linear extension is a list of the labels that starts at some minimal element and works up the poset respecting the order relations: if $i < j$ in the poset, then i comes before j in list.

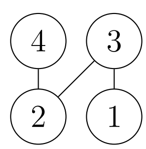


Figure 2.8: A labeled poset of size 4

A *descent* of a permutation $\pi = (\pi_1, ..., \pi_n)$ of $\{1, 2, 3, ..., n\}$ is a position i such that $\pi_i > \pi_{i+1}$. This is denoted $Des(pi)$. For example, the descents of $\pi = (2, 3, 1, 5, 4)$

are 2 and 4. Given a permutation π of $\{1, 2, 3, ..., n\}$, we define a *descent composition* $\text{co}(\pi)$ by

$$\text{co}(\pi) = (d_1, d_2 - d_1, d_3 - d_2, ..., d_k - d_{k-1}, n - d_k)$$

where the descents of π are at positions $d_1 < d_2 < \cdots < d_k$. Take for example the permutation $\pi = (2, 3, 1, 5, 4)$. We know that $d_1 = 2$, $d_2 = 4$, and $n = 5$. From the definition this gives us that $\text{co}(\pi) = (2, 4 - 2, 5 - 4) = (2, 2, 1)$. In practice this is determined by counting the runs of strictly increasing numbers in π. For example we would determine the descent set of $(2, 3, 1, 5, 4)$ by recognizing the first run $(2, 3)$ is of length 2, the second run $(1, 5)$ is of length 2, and the third and final run is just (4), of length 1.

The following theorem will drive many of our proofs, and is the simplest method of computing the F-basis expansion of a given poset's generating function.

Theorem 2.4.9. *[Sta71, Sta72, Ges84] For labeled poset (P, ω),*

$$K_{(P,\omega)} = \sum_{\pi \in \mathcal{L}(P,\omega)} F_{co(\pi)}$$

where the sum is over all linear extensions π of (P, ω).

For example the poset (P, ω) in Figure 2.5 whose partition generating function we computed earlier has linear extensions $(2, 1, 3)$ and $(2, 3, 1)$. The compositions of these are $(1, 2)$ and $(2, 1)$, confirming that $K_{(P,\omega)} = F_{12} + F_{21}$. As for the Poset (Q, τ) in Figure 2.8, We would perform the calculation as follows:

Linear Extension	Composition	Fundamental Basis Element
$(1, 2, 3, 4)$	(4)	F_4
$(1, 2, 4, 3)$	$(3, 1)$	F_{31}
$(2, 1, 3, 4)$	$(1, 3)$	F_{13}
$(2, 1, 4, 1)$	$(1, 2, 1)$	F_{121}
$(2, 4, 1, 3)$	$(2, 2)$	F_{22}

Thus we have that $K_{(Q,\tau)} = F_4 + F_{31} + F_{13} + F_{121} + F_{22}$. Especially in the second case, the method of Theorem 2.4.9 can be *much* simpler than guessing every configuration of P-partition and building up the M-basis.

2.5 Involutions on labeled posets

Additionally, there are elementary transformations that can be applied to posets that lead to trivial implications on equality. For a labeled poset (P, ω), we take $\overline{(P, \omega)}$ to be the transformation that changes all strict edges to weak and vice versa. We take $(P, \omega)^*$ to be the transformation that rotates a poset 180° while preserving strict and weak edges. We will refer to the map that sends (P, ω) to $\overline{(P, \omega)}$ (resp. $(P, \omega)^*$) as the *bar* (resp. *star*) *involution.* An example of the effects of these involutions on the Hasse diagram of a poset can be seen in Figure 2.9. The following Lemma from

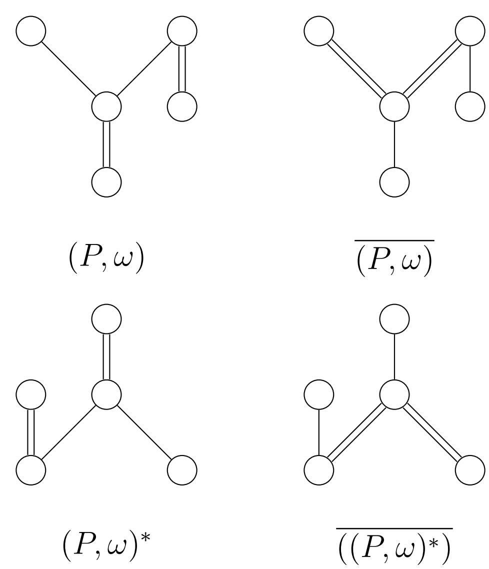


Figure 2.9: A labeled poset (P, ω) and its star, bar, and combined star and bar involution.

[MW] will allow us to greatly lessen the number of relations we need to explain.

Lemma 2.5.1. *[MW] Let (P, ω) be a labeled poset. Then we have:*

(a) the descent compositions of the linear extensions of $\overline{(P, \omega)}$ are the complements of the descent compositions of the linear extensions of (P, ω);

(b) the descents compositions of the linear extensions of $(P, \omega)^*$ are the reverses of the descent compositions of the linear extensions of (P, ω).

We deduce from this a result analogous to [MW, Prop. 3.7]:

Proposition 2.5.2. *For labeled posets (P, ω) and (Q, τ), the following are equivalent:*

- $(P, \omega) \leq_F (Q, \tau)$
- $(P, \omega)^* \leq_F (Q, \tau)^*$
- $\overline{(P, \omega)} \leq_F \overline{(Q, \tau)}$
- $\overline{((P, \omega)^*)} \leq_F \overline{((Q, \tau)^*)}$

This will allow us to study several different relations through the analysis of one.

Remark 2.5.3. Because of Proposition 2.4.7, we know that the relations above in Proposition 2.5.2 also hold with respect to F-support, M-positivity, and M-support.

2.6 Orders on Sequences

This section contains necessary definitions of orders that we can put on sequences of numbers. The definitions here are vital for several of the conditions for Chapter 3.

Definition 2.6.1. Given two sequences $p = (p_1, p_2, ...p_a)$ and $q = (q_1, q_2, ..., q_b)$, the sequence p is less than q in *dominance* order if $\sum_{i=1}^{k} p_i \leq \sum_{j=1}^{k} q_j$ for all $k \geq 1$, where $p_i = 0$ if $i > a$ and $q_j = 0$ if $j > b$. We denote this as $p \leq_{dom} q$.

Definition 2.6.2. Given two sequences $p = (p_1, p_2, ...p_a)$ and $q = (q_1, q_2, ..., q_b)$, the sequence p is less than q in *lexicographic* order if $p_i < q_i$ for the first i where $p_i \neq q_i$. We denote this $p \leq_{lex} q$.

Take the sequences $(1, 6, 1)$, $(3, 2, 2, 2)$, and $(3, 2, 4, 1, 1)$. We know that $(3, 2, 4, 1, 1) >_{lex} (3, 2, 2, 2) >_{lex} (1, 6, 1)$. We know that $(3, 2, 4, 1) >_{lex} (3, 2, 2, 2)$ because the first element between $(3, 2, 4, 1, 1)$ and $(3, 2, 2, 2)$ that differs is the third one and $4 > 2$. We know that $(1, 6, 1)$ differs from both of the others in the first slot, and since $3 > 1$ we get the relation above. Lexicographic ordering gives a total order on sequences, and two are going to either be equal or comparable in lexicographic ordering.

Take those same three sequences and look at them in dominance order. We know that $(3, 2, 2, 2) \not\leq_{dom} (1, 6, 1)$ because $1 < 3$, however we also have that $(1, 6, 1) \not\leq_{dom} (3, 2, 2, 2)$ because $3 + 2 < 1 + 6$. Thus there is no dominance relation between the two sequences. Looking at $(3, 2, 2, 2)$ and $(3, 2, 4, 1, 1)$, we get that $(3, 2, 2, 2) \leq_{dom} (3, 2, 4, 1, 1)$ because

1) $3 \leq 3$,
2) $3 + 2 \leq 3 + 2$,
3) $3 + 2 + 2 \leq 3 + 2 + 4$,
4) $3 + 2 + 2 + 2 \leq 3 + 2 + 4 + 1$,
5) and finally $3 + 2 + 2 + 2 \leq 3 + 2 + 4 + 1 + 1$.

The following lemma relates these two ordering systems.

Lemma 2.6.3. *For sequences* $p = (p_1, p_2, ...p_a)$ *and* $q = (q_1, q_2, ..., q_b)$*, if* $p \leq_{dom} q$ *then* $p \leq_{lex}$.

Proof. If the sequences are equal this is trivial. If two elements $p = (p_1, p_2, ...p_a)$ and $q = (q_1, q_2, ..., q_b)$ are related in dominance order such that $p <_{dom} q$, we know that the lowest index i in which the element $p_i \neq q_i$ must be such that

$$\sum_{k=0}^{i} p_k < \sum_{l=1}^{i} q_l.$$

However since $p_j = q_j$ for all $j < i$, we know that $p_i < q_i$ and therefore $p \leq_{lex} q$. □

Chapter 3

Necessary Conditions

This section will focus on necessary conditions for F-positivity. Our first of two main results is for naturally labeled posets, and our second is on a construction called the *jump sequence* of a poset.

3.1 Weak and Strict Edges

Lemma 3.1.1. *If* $(P, \omega) \leq_F (Q, \tau)$ *and* (P, ω) *only has weak edges, then so does* (Q, τ).

Proof. We examine the linear extensions of (P, ω), and seek to show by induction that $(1, 2, ..., |P|)$ is a linear extension of P. If $|P| = 1$, this is trivial. Say then that $(1, 2, ..., k)$ is a linear extension of P_k for any labeled poset P_k such that $|P_k| = k$. Take then a poset P_{k+1} such that $|P_{k+1}| = k+1$. Impose a labeling such that all edges are weak. Because all edges are weak, we know that the element labeled $k + 1$ must be a maximal element. Remove that element, and we are left with a poset that has k elements and all weak edges. By our induction hypothesis, we know that $(1, 2, ..., k)$ is a valid linear extension for this poset. Since our deleted element was maximal, we know that for P_{k+1} that $(1, 2, ..., k, k+1)$ is a valid linear extension. Thus by induction we know that $(1, 2, ..., |P|)$ is a linear extension of (P, ω). Since $(1, 2, ..., |P|)$ is strictly increasing, its descent composition is simply $(|P|)$, and therefore we know that $F_{|P|}$ is an element of the support of (P, ω).

Suppose to the contrary that the labeled poset (Q, τ) has at least one strict edge. Then there exist elements $p, q \in Q$ such that $p < q$ in Q however $\tau(p) > \tau(q)$. This relation must be respected in any linear extension of (Q, τ). Thus there is at least one descent in every linear extension of (Q, τ), as $\tau(p)$ must appear before $\tau(q)$. Therefore $F_{|Q|} = F_{|P|}$ is not in the support of (Q, τ). We conclude then that $(P, \omega) \not\leq_F (Q, \tau)$, a contradiction. Thus (Q, τ) contains no strict edges. $\square$

Applying Proposition 2.5.2, we get the following corollary:

Corollary 3.1.2. *If $(P, \omega) \leq_F (Q, \tau)$ and (P, ω) only has strict edges, then so does (Q, τ).*

3.2 Ordering on the Jump

This section introduces the jump a labeled poset, and the resulting necessary conditions that must be fulfilled for two labeled posets to be comparable. The following definition will assist us in defining the jump.

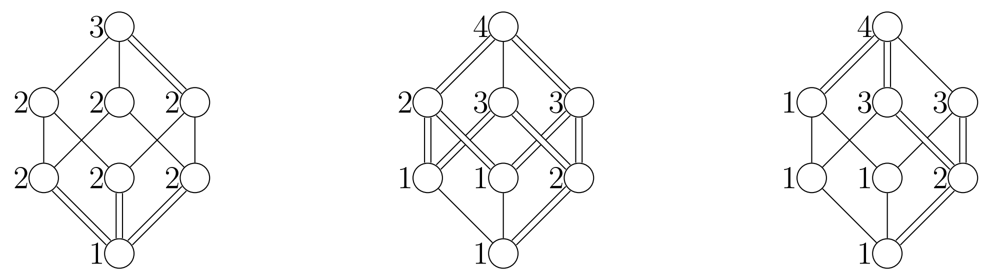
Definition 3.2.1. A *chain* is a sequence of elements of a poset $x_1, x_2, x_3, ..., x_k$ where $x_1 <_P x_2 <_P x_3 <_P \cdots <_P x_k$. A chain is a *saturated chain* if all of the relations $x_i <_P x_{i+1}$ are cover relations.

Definition 3.2.2. The *jump*(p) of an element p of a labeled poset (P, ω) is the maximum number of strict relations in a saturated chain from the element to a minimal element of P. The *jump sequence* of (P, ω), denoted jump(P, ω), is defined by

$\text{jump}(P,\omega) = (j_0, ..., j_k)$, where j_i is the number of elements of elements of jump i, and k is the maximum jump of an element of (P,ω).

It will be useful (particularly in Chapter 7) to notate $\overline{\text{jump}}(P,\omega)$ as the jump of the poset taken after the bar-involution. Realize this is the same as the jump if we count weak relations instead of strict.

Alternatively, one can easily find the jump of a poset by considering the (P,ω)-partition where each number assigned is the least one possible. For example the jump sequences of the posets (P,ω), (Q,τ), and (R,ρ) in Figure 3.1 are $(1,6,1)$, $(3,2,2,1)$,



(P,ω) (Q,τ) (R,ρ)

Figure 3.1: Three posets with the partition that gives the jump sequence for each.

and $(4,1,2,1)$ respectively.

The *star-jump* of an element and the *star-jump sequence* of a poset is defined similarly, but with saturated chains to a *maximal* element. This is also the same as the jump under the star involution. Since all relations must hold under this involution, we know that all necessary conditions we place on the jump sequence will have a corollary in the star-jump sequence.

Observe further that the jump(p) of an element p is the minimum value of $f(p)$ over all (P,ω)-partitions f. Therefore for any (P,ω)-partition f, the number of elements p with $f(p) \leq i$ cannot exceed

$$\text{jump}_1(P,\omega) + \text{jump}_2(P,\omega) + \cdots + \text{jump}_i(P,\omega). \tag{3.1}$$

This relates nicely to Definition 2.6.1, and leads to our next lemma:

Lemma 3.2.3. *The M-support of (P, ω) contains jump(P, ω), which is the maximum element of the M-support under dominance order.*

Proof. Let the composition $\alpha = (\alpha_1, ..., \alpha_k)$ be an element of the M-support of $K_{(P,\omega)}$, and let f be a (P, ω)-partition that contributes $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ to $K_{(P,\omega)}$. Then for $1 \leq i \leq k$, we have $\alpha_1 + ... + \alpha_i$ equals the number of elements of (P, ω) that f maps to the elements of $\{1, 2, ..., i\}$. By the observation in (3.1), we know that $\alpha_1 + ... + \alpha_i \leq \text{jump}_1(P, \omega) + ... + \text{jump}_i(P, \omega)$. Thus we know that $\alpha \leq_{dom} \text{jump}(P, \omega)$. It also follows directly from our observation after Definition 3.2.2 that jump(P, ω) is in the M-support of (P, ω), as required. □

This leads to the following necessary condition, which is our main necessary condition involving the jump sequence.

Corollary 3.2.4. *If* $\text{supp}_M(P, \omega) \subseteq \text{supp}_M(Q, \tau)$*, then jump$(P, \omega) \leq_{dom}$ jump(Q, τ).*

Proof. By Lemma 3.2.3, we have that jump(P, ω) must be an element of the M-support of $K_{(Q,\tau)}$. Thus, again by Lemma 3.2.3 we have that jump$(P, \omega) \leq_{dom}$ jump(Q, τ). □

Note that we assume the condition that $\text{supp}_M(P, \omega) \subseteq \text{supp}_M(Q, \tau)$, which is the weakest condition we require. By Proposition 2.4.7 we can take any of the other three relations there as our hypothesis.

Corollary 3.2.5. *If* $\text{supp}_M(P, \omega) \subseteq \text{supp}_M(Q, \tau)$*, then jump$(P, \omega) \leq_{lex}$ jump(Q, τ).*

Proof. By Corollary 3.2.4 and Lemma 2.6.3 we get jump$(P, \omega) \leq_{lex}$ jump(Q, τ). □

Similarly, any formulation we would make for the star-jump sequence can be handled under the same circumstances with the star involution. Consider the posets from

Figure 3.1. We have stated before that their jump sequences are $(1,6,1)$, $(3,2,2,1)$, and $(4,1,2,1)$ respectively. We can use this to tell whether pairs of those three posets are related in the F-positivity order. We know that $(R,\rho) \not\leq_F (Q,\tau)$ and $(Q,\tau) \not\leq_F (P,\omega)$ because $(4,1,2,1) >_{lex} (3,2,2,1) >_{lex} (1,6,1)$. We also know that $(P,\omega) \not\leq_F (Q,\tau)$ and $(P,\omega) \not\leq_F (R,\rho)$ because $(1,6,1) \not\leq_{dom} (3,2,2,1)$ and $(1,6,1) \not\leq_{dom} (4,2,1,1)$. When examining whether or not $(Q,\tau) \leq_F (R,\rho)$, we see that $(3,2,2,1) \leq_{lex} (4,1,2,1)$ and $(3,2,2,1) \leq_{dom} (4,1,2,1)$. One might be inclined then to guess that $(Q,\tau) \leq_F (R,\rho)$, but this is not the case. It is this property that makes these conditions necessary and not sufficient.

The following series of claims are other ways to quickly get necessary conditions that are also based on elements of the M-support.

Lemma 3.2.6. *If* $\mathrm{supp}_M(P,\omega) \subseteq \mathrm{supp}_M(Q,\tau)$*, then the largest number of strict connections in a saturated chain in* (P,ω) *must be greater than or equal to the largest number of strict connections in a saturated chain in* (Q,τ)*.*

Proof. We have by Corollary 3.2.4 that $\mathrm{jump}(P,\omega) \leq_{dom} \mathrm{jump}(Q,\tau)$. Since the elements of $\mathrm{jump}(P,\omega)$ and $\mathrm{jump}(Q,\tau)$ both sum to $|P| = |Q|$, we know that the length of the jump sequence of (P,ω) must be greater than or equal to that of (Q,τ). The length of a jump sequence is the largest value of $\mathrm{jump}(p)$ for $p \in P$. Therefore the length of the jump sequence is also the largest number of strict connections in a saturated chain from an element down to a minimal element. We conclude that the largest number of strict connections in a saturated chain in (P,ω) must be greater than or equal to the largest number of strict connections in a saturated chain in (Q,τ). □

Corollary 3.2.7. *If* $\mathrm{supp}_M(P,\omega) \subseteq \mathrm{supp}_M(Q,\tau)$*, then the largest number of weak connections in a saturated chain in* (P,ω) *must be greater than or equal to the largest number of weak connections in a saturated chain in* (Q,τ)*.*

Proof. Apply the bar operation to both (P,ω) and (Q,τ), from Remark 2.5.3 we know $\text{supp}_M\overline{(P,\omega)} \subseteq \text{supp}_M\overline{(Q,\tau)}$. Then apply Lemma 3.2.6, and recall that the number of strict edges edges in a chain in the $\overline{(P,\omega)}$ (resp. $\overline{(Q,\tau)}$) is the same as the number of weak edges in a chain in (P,ω) (resp. (Q,τ)). □

Definition 3.2.8. A sub-poset (S,σ) inside of (P,ω) is *convex* if $y \in S$ if $x \leq_P y \leq_P z$ and $x, z \in P$.

Definition 3.2.9. The *weak width* of a labeled poset (P,ω), denoted weakwidth(P,ω), is the number of elements in the largest convex subposet of (P,ω) that contains only weak edges. See Figure 3.2.

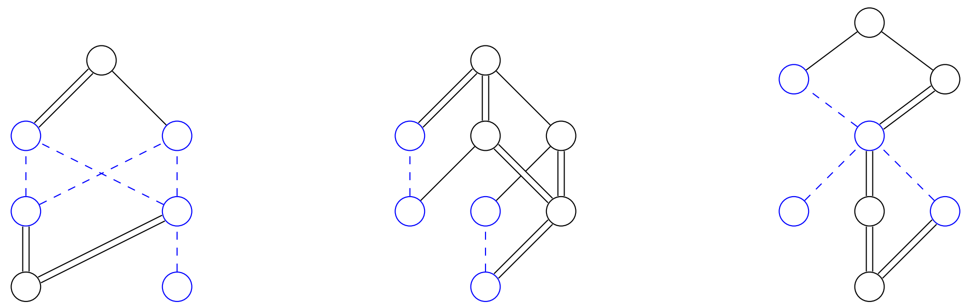


Figure 3.2: Examples of posets with their (not always unique) largest convex subposet containing only weak dashed edges colored in blue

Observe that the weak width is also the largest number of elements that can be mapped to the same value by a (P,ω)-partition.

Corollary 3.2.10. *Suppose* (P,ω) *and* (Q,τ) *are labeled posets. If* $\text{supp}_M(P,\omega) \subseteq \text{supp}_M(Q,\tau)$*, then weakwidth*$(P,\omega) \leq$ *weakwidth*(Q,τ)*.*

Proof. Let k be the weak width of (P,ω). By definition of weak width, there exists an element α in $\text{supp}_M(P,\omega)$ that itself has an element equaling k. Since $\text{supp}_M(P,\omega) \subseteq$

$\mathrm{supp}_M(Q, \tau)$, we know that (Q, τ) contains a convex subposet of size at least k that contains all weak elements. Therefore $\mathrm{weakwidth}(Q, \tau) \geq k = \mathrm{weakwidth}(P)$. □

Chapter 4

Adding, Deleting, and Linear Extensions

This section focuses on the relationships that result from deletion of an edge to an existing labeled poset, preceded potentially by the addition of what we will define as a "redundant edge."

Note that if $\mathcal{L}(P,\omega) \subseteq \mathcal{L}(Q,\tau)$, then $(P,\omega) \leq_F (Q,\tau)$ by Theorem 2.4.9. This will form the basis for many of our sufficient conditions for $(P,\omega) \leq_F (Q,\tau)$.

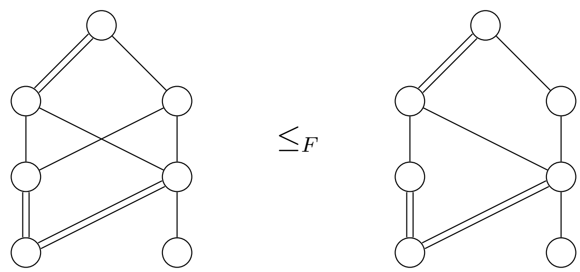
4.1 Deleting Edges

The simplest way that we obtain linear extension containment is by simply deleting an edge from the Hasse diagram. Thus we get the following proposition, which this chapter as a whole will seek to generalize:

Proposition 4.1.1. *If (Q,τ) is obtained from (P,ω) by deleting an edge from the Hasse diagram of (P,ω), then $\mathcal{L}(P,\omega) \subseteq \mathcal{L}(Q,\tau)$.*

Proof. Identify each element in P with the corresponding label in Q. Since (Q,τ) is simply (P, ω) without an edge, we know that ω and τ can label each corresponding pair of elements of P and Q with the same number, essentially getting the same labeling. Take a linear extension γ of (P, ω). We know that every edge of (P, ω) is in (Q, τ), and that γ is a linear extension in (P, ω) (and therefore (Q, τ)) if and only if γ respects all of the edges in the poset. Thus we know that γ is also a linear extension of (Q, τ). Since γ was arbitrary we know that $\mathcal{L}(P\omega) \subseteq (Q, \tau)$. □

As an example, we can explain poset relations such as



(P, ω) (Q, τ)

by means of Proposition 4.1.1.

4.2 Augmented Diagrams and Labels

Definition 4.2.1. An *augmented diagram* $(P, \omega)^{+E}$ of a labeled poset (P, ω) consists of the Hasse diagram of (P, ω) along with a set E of edges (x, y) such that $x <_P y$ but (x, y) is not a covering relation. Each such additional edge is defined as strict or weak independent of ω.

Examples of augmented diagrams can be seen in Figure 4.1.

Definition 4.2.2. Given an augmented diagram $(P, \omega)^{+E}$ form a directed graph D by orienting weak edges upwards and strict edges downwards. Then we say $(P, \omega)^{+E}$ has a *bad cycle* if it has any directed cycles.

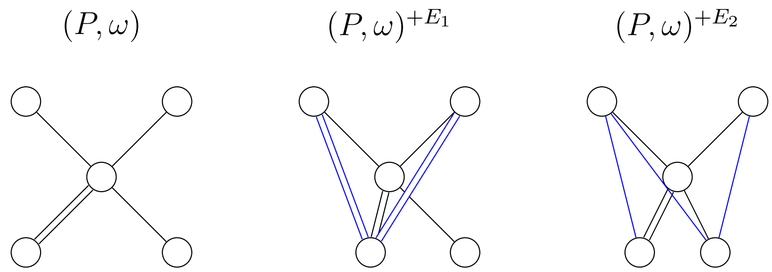


Figure 4.1: A poset (P, ω) with two different augmentations.

For examples of what bad cycles can look like, see Figure 4.2. We will restrict ourselves in this thesis to augmented diagrams that do not contain any bad cycles, primarily because of the Lemma below.

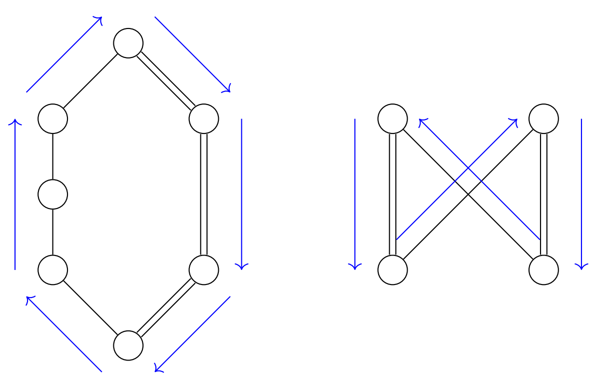


Figure 4.2: Examples of posets with a bad cycle

Lemma 4.2.3. *There exists a labeling ω' of $(P, \omega)^{+E}$ that respects all of the strict and weak conditions on the edges including those for E if and only if $(P, \omega)^{+E}$ has no bad cycles.*

Proof. First assume that there exists a labeling ω' for $(P, \omega)^{+E}$ that respects all of the strict and weak edges. Assume for contradiction that $(P, \omega)^{+E}$ contains a bad cycle, whose elements in order of the cycle are $x_1, x_2, ..., x_k$. If $(P, \omega)^{+E}$ has a labeling, then each element x_i of this bad cycle must have a label associated with it. Notice however that as we have defined the direction of these edges, an edge in the directed graph $x_i \to x_{i+1}$ implies that $\omega'(x_i) < \omega'(x_{i+1})$. This means that in our bad cycle, $\omega'(x_1) < \omega'(x_2) < \cdots < \omega'(x_k) < \omega'(x_1)$. This is of course a contradiction, as $\omega'(x_1)$ cannot be strictly greater than itself. Thus we know that there can be no bad cycle in $(P,\omega)^{+E}$.

Now assume that $(P,\omega)^{+E}$ contains no bad cycles. We will construct an explicit labeling of $(P,\omega)^{+E}$ that respects all of the new edges that were written in. As before we can form a directed graph from $(P,\omega)^{+E}$ where the weak edges are oriented upwards and the strict edges are oriented downwards. Form a new all weak-edged poset Q defined by $a \leq_Q b$ if there is a directed path from a to b. See Figure 4.3 for an example. We represent Q as an augmented diagram whose set of edges equals that of $(P,\omega)^{+E}$ but possibly with new directions. We must check that Q is a valid poset.

We know that Q is reflexive and transitive trivially.

Say that Q is not anti-symmetric. This implies the existence of a, b such that $a \neq b$ however $a \leq_Q b$ and $b \leq_Q a$. This means however that there exists a directed path from a to b and b to a, which implies the existence of a bad cycle. Thus we know Q must be anti-symmetric.

It is clear then that Q can be naturally labeled, as all of the edges are weak. We can apply this labeling to the elements of $(P,\omega)^{+E}$ and it will respect all strict and weak edges, because if $a < b$ in $(P,\omega)^{+E}$ is a strict edge then $a > b$ in Q, and if $a < b$ in $(P,\omega)^{+E}$ is a weak edge then $a < b$ in Q. By construction we have that a labeling of $(P,\omega)^{+E}$ that respects the strictness and weakness of all edges. □

From Definition 4.2.1 and 4.2.3, it makes sense to refer to edges (a, b) that are added to augmented diagrams and that don't create bad cycles as *redundant edges*. This is because their relation (a, b) is already implied by the edges in the Hasse

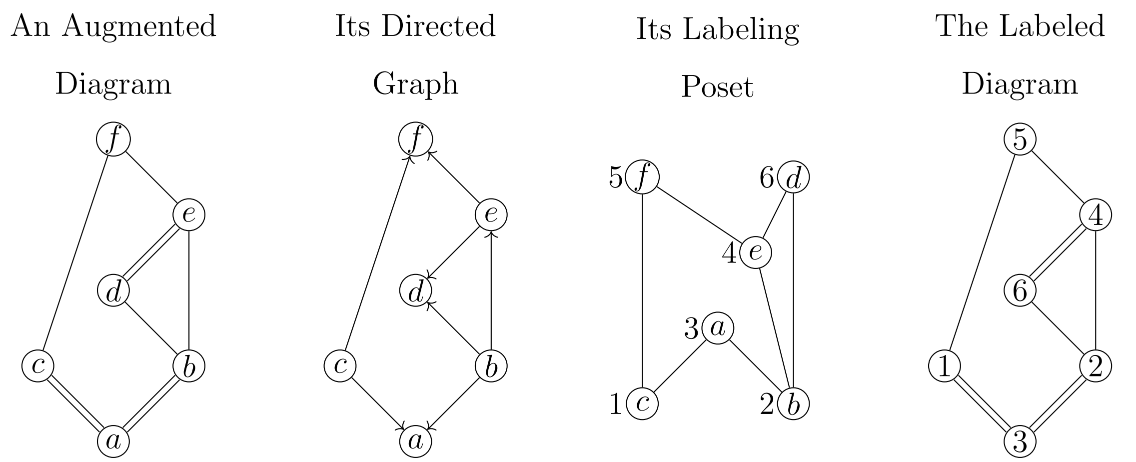


Figure 4.3: An example of the process of labeling an augmented diagram as described in Lemma 4.2.3.

diagram, and the validity of their strictness and weakness (whichever that may be) is implied by whether or not that same strictness or weakness would create a bad cycle.

4.3 Less-Than Sets

Definition 4.3.1. The *less-than set* of labeled poset (P, ω) is the set of label relations $S_<(P, \omega) = \{(\omega(a), \omega(b)) \colon a <_P b\}$. This is also called the *transitive closure* of the cover relations.

This section will focus on the deletion of elements from a less-than set, and what that may imply about poset relations. For simplicity's sake, for the remainder of the section we will identify each element of a poset with its label, so that $\omega^{-1}(a) \leq_P \omega^{-1}(b)$ can instead be written as $a \leq_P b$.

Definition 4.3.2. We say that a linear extension (or any permutation) ρ *violates* an element (a, b) of a less-than set if b appears before a in ρ.

Note that if a permutation ρ violates any $(a, b) \in S_<(P, \omega)$, then ρ cannot be a linear extension of (P, ω). Moreover, by the definition of linear extension (Definition 2.4.8), if there are no violations of $S_<(P, \omega)$ in ρ, then ρ is a linear extension of (P, ω).

Lemma 4.3.3. *Say that (P, ω) is a labeled poset with $a <_P b$. Then $S_<(P, \omega) \setminus (a, b)$ is a less-than set for another labeled poset Q if and only if $a <_P b$ is a cover relation.*

Proof. We know that regardless of what type of relation (a, b) is, $S_<(P, \omega) \setminus (a, b)$ will still preserve reflexivity and antisymmetry. Thus we must check transitivity.

Say first that (a, b) is a covering relation. Let us take then $(x, y), (y, z) \in S_<(P, \omega) \setminus (a, b)$. Since $S_<(P, \omega)$ is transitive we know that $(x, z) \in S_<(P, \omega)$. Clearly $x \leq_P z$ is not a cover relation in (P, ω), thus we know that $(x, z) \in S_<(P, \omega) \setminus (a, b)$. Thus the set of relations preserves transitivity, and represents a poset. By construction the same labeling ω can be used to label the new poset, and we have that $S_<(P, \omega) \setminus (a, b) = S_<(Q, \tau)$ for some labeled poset (Q, τ).

Now say that (a, b) is not a cover relation. This means that in (P, ω) there exists a c such that $a <_P c <_P b$. We know however then that $(a, c), (a, b) \in S_<(P, \omega) \setminus (a, b)$, while (a, b) is clearly not. Thus $S_<(P, \omega) \setminus (a, b)$ is not transitive, and therefore does not represent a poset. □

Note that repeated application of Lemma 4.3.3 can result in a wide variety of posets. For example the posets (Q, τ) and (R, ρ) in Figure 4.4 are obtained by deleting cover relations from the less-than set of (P, ω) in the following way:

$$S_<(Q, \tau) = S_<(P, \omega) \setminus \{(2, 6)\} \setminus \{(2, 5)\} \setminus \{(4, 5)\}$$

$$S_<(R, \rho) = S_<(P, \omega) \setminus \{(2, 5)\} \setminus \{(2, 6)\} \setminus \{(4, 6)\} \setminus \{(3, 6)\}.$$

Note how some of the relations deleted are not cover relations in $S_<(P, \omega)$. They are however cover relations once a *previous* deletion has been made. For example, the

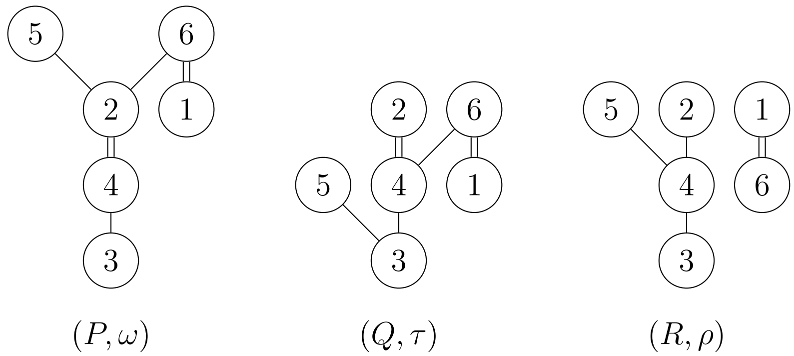


Figure 4.4: Three labeled posets (P, ω), (Q, τ), and (R, ρ) where both(R, ρ) and (Q, τ) are obtained from (P, ω) by deleting some cover relations.

relation $(4, 6)$ that was deleted to make (R, ρ) is not a cover relation in $S_<(P, \omega)$, however it is a cover relation in the poset obtained from first deleting $(2, 6)$. This is what we mean by deleting a *series* of cover relations.

We will connect the less-than sets with the set of linear extensions, and to do so we need a preliminary lemma.

Lemma 4.3.4. *For poset (P, ω), if $a \not\leq_P b$ then there is a linear extension ρ such that b appears before a in ρ.*

Proof. If $a >_P b$, then by definition *every* linear extension of (P, ω) must have b come before a. Thus we only need to check the case where a is not related to b.

Partition all of the elements of (P, ω) into two sets as follows: the first set A is all of the elements $x \neq a, b$ of (P, ω) such that $x \not>_P a$ and $x \not>_P b$, and the second set B is all of the remaining elements $y \neq a, b$ such that $y >_P a$ or $y >_P b$. Build ρ as follows:

$$\rho = ((\text{linear extension of } x \in A), b, a, (\text{linear extension of } y \in B)).$$

Take $p \in (P, \omega) \setminus \{a, b\}$. If $p \notin A$ then $p >_P a$ or $p >_P b$, so $p \in B$. Thus we know that ρ does contain all of the elements of (P, ω). Similarly, p cannot be both greater than and not greater than a or b, so we know that $A \cap B = \emptyset$.

We know that within the linear extensions of A and B all of the relations will hold. The sets are also defined such that there will be no conflicts in ρ where an element greater than b or a occurs before them (in A) or an element less than b or a occurs after them (in B). This is because A (everything before) is *defined* as containing all elements less than a or b, so no such element is in B. It thus remains to be seen that every element in A is not greater than any element in B. Take $x \in A$ and $y \in B$, and say for contradiction that $x >_P y$. Then $x >_P y >_P a$ or $x >_P y >_P b$. In either case this means that $x \in B$, but $A \cap B = \emptyset$ so there cannot exist such an x.

Thus we know that ρ is a valid linear extension of (P, ω), and by construction we have our claim. □

The next lemma seeks to relate containment of these less-than sets to containment of sets of linear extensions.

Lemma 4.3.5. *Take two labeled posets P and Q. Then $\mathcal{L}(P, \omega) \subseteq \mathcal{L}(Q, \tau)$ for some valid (Definition 2.2.2) ω and τ if and only if $S_<(P, \omega) \supseteq S_<(Q, \tau)$ for the same ω and τ.*

Proof. Say first that $S_<(P, \omega) \supseteq S_<(Q, \tau)$. Now take any $\rho \in \mathcal{L}(P, \omega)$. We know that ρ is a linear extension of a poset Q if it obeys all of the relations in Q. We know that all of the relations in Q are also relations in P, thus we know that if ρ obeys all relations in P that it will obey all relations in Q. Thus we have that $\rho \in \mathcal{L}(Q, \tau)$.

Now say that $\mathcal{L}(P, \omega) \subseteq \mathcal{L}(Q, \tau)$. We seek to show that $S_<(P, \omega) \supseteq S_<(Q, \tau)$. For contradiction assume that there is a $\gamma \in S_<(Q, \tau)$ such that $\gamma \notin S_<(P, \omega)$. Let $\gamma = (a, b)$. Since $\gamma \notin S_<(P, \omega)$, we know by Lemma 4.3.4 that there must be a linear extension $\rho \in \mathcal{L}(P, \omega)$ such that b appears before a in ρ. This means however that ρ violates γ, which is contained within $S_<(Q, \tau)$. This means that $\rho \notin \mathcal{L}(Q, \tau)$. This contradicts our assumption that $\mathcal{L}(P,\omega) \subseteq \mathcal{L}(Q,\tau)$, thus we know that there cannot exist such a γ, and therefore $S_<(P,\omega) \supseteq S_<(Q,\tau)$. □

The following lemma will establish our connection to the augmented diagrams.

Lemma 4.3.6. *For labeled posets (Q,τ) and (P,ω), we have $S_<(P,\omega) \supseteq S_<(Q,\tau)$ if and only if $S_<(Q,\tau)$ can be obtained from $S_<(P,\omega)$ by a series of deletions of cover relations.*

Proof. If $S_<(Q,\tau)$ can be obtained from $S_<(P,\omega)$ by a series of deletions of cover relations, then it is clear that $S_<(Q,\tau) \subseteq S_<(P,\omega)$.

Assume that $S_<(P,\omega) \supset S_<(Q,\tau)$. We seek to show that there must always be a cover relation $a < b$ of (P,ω) such that $a \nleq_Q b$. Say for contradiction that all cover relations of (P,ω) were contained within $S_<(Q,\tau)$. We know however that cover relations generate the entire poset, and therefore $S_<(P,\omega) \subseteq S_<(Q,\tau)$, a contradiction. Thus $S_<(P,\omega) \setminus S_<(Q,\tau)$ contains a cover relation $(a,b) \in S_<(P,\omega)$. Delete this cover relation from $S_<(P,\omega)$, and by Lemma 4.3.3 you have a (P',ω') such that $S_<(P',\omega') \supseteq S_<(Q,\tau)$.

Repeated application of this process means that eventually $S_<(Q,\tau)$ will be obtained by a deletion of a cover relation, and we have our claim. □

The following corollary is a corollary to the previous two lemmas.

Corollary 4.3.7. *For labeled posets (P,ω) and (Q,τ), $\mathcal{L}(P,\omega) \subseteq \mathcal{L}(Q,\tau)$ if and only if $S_<(Q,\tau)$ can be obtained by a sequence of deletions of cover relations from $S_<(P,\omega)$.*

Proof. This follows directly from in Lemmas 4.3.5 and 4.3.6. □

In summary we have a necessary and sufficient condition in terms of the less-than sets for inequalities among posets that are due to linear extension containment. However, our real goal is such a condition in terms of adding redundant edges and deleting edges; this is the focus of the next section.

4.4 Linear Extension Containment

The previous methods for determining linear extension containment, as appearing in Proposition 4.1.1 and Corollary 4.3.7, each have their different advantages and disadvantages. The less-than sets provide the benefit of the converse also being true, however writing down the whole less-than set is an arduous task. The deletion method with Hasse diagrams is sufficiently visual that it can be checked easily by eye in many cases, but is far from the only way to get linear extension containment. The following theorem combines the best of both methods; it is our main result of the chapter. Recall valid alternative labelings, as defined in Definition 2.2.2.

Theorem 4.4.1. *Let (P, ω) and (Q, τ) be labeled posets. The following are equivalent:*

(1) $\mathcal{L}(P, \omega') \subseteq \mathcal{L}(Q, \tau')$ *for some valid alternative labelings* ω' *and* τ'*.*

(2) $S_<(P, \omega') \supseteq S_<(Q, \tau')$ *for some valid alternative labelings* ω' *and* τ'*.*

(3) (Q, τ) *is obtained from* (P, ω) *by adding redundant edges and then deleting edges from the augmented diagram.*

The following Figure 4.5 shows how this theorem might be applied to a Hasse diagram in order to get a relation.

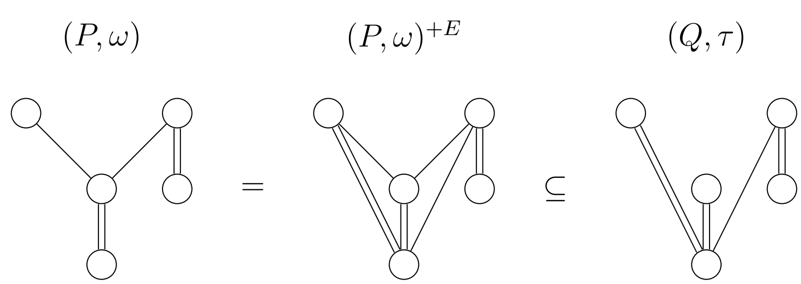


Figure 4.5: An example of how this addition and deletion can be used to show that one poset's set of linear extensions are contained within another's

Proof of Theorem 4.4.1. Much of this proof has already been completed. Look first at (1)⇔(2), which is true by Lemma 4.3.5 with ω' and τ' playing the roles of ω and τ respectively. To complete the theorem we will show here that (2)⇔(3).

Assume (2): $S_<(P,\omega') \supseteq S_<(Q,\tau')$ for some alternative labelings ω' and γ'. Look at the Hasse diagrams for (P,ω') and (Q,τ'), and write in every redundant edge implied by the labels. We know that all relations in (Q,τ') are also in (P,ω'), thus all edges redundant or otherwise in this augmented diagram of (Q,τ') are in the augmented diagram for (P,ω'). This means that we can delete some set of redundant and non-redundant edges from the augmented diagram for (P,ω') to get all of and only the edges of the augmented diagram for (Q,τ'). From there we can remove all of the redundant edges from what remains to get the regular Hasse diagram for (Q,τ'). Since by definition τ' imparts the same strict and weak cover relations as τ on Q, we know that this is also the Hasse diagram for (Q,τ). Similarly the Hasse diagrams for (P,ω) and (P,ω') are also identical. Thus we have that (Q,τ) is obtained from (P,ω) by adding redundant edges and then deleting edges from the augmented diagram.

Now assume (3): (Q,τ) is obtained from (P,ω) by adding redundant edges (without bad cycles) and then deleting edges from the augmented diagram $(P,\omega)^{+E}$. By Lemma 4.2.3 we define ω' to be the labeling of $(P,\omega)^{+E}$ that respects all of the redundant edges as well. Now to (P,ω) add *all* redundant edges A_P with strictness/weakness determined by ω', giving us $(P,\omega')^{+A_P}$. Realize that this is a graphical representation of $S_<(P,\omega')$, the transitive closure of (P,ω').

Our assumption is that (Q,τ) is achieved by deleting edges from $(P,\omega)^{+E}$. Since $(P,\omega')^{+A_P}$ is $(P,\omega)^{+E}$ with even more redundant edges added in, we know that the Hasse diagram for (Q,τ) can also be achieved by deleting edges from $(P,\omega')^{+A_P}$. Now take $\tau' = \omega'$. Since all cover relations of (Q,τ) are in the diagram for $(P,\omega')^{+A_P}$, we know that τ' *must* be a valid alternative labeling for (Q,τ). Now add all redundant edges A_q to (Q, τ) with strictness and weakness determined by τ' to get the augmented diagram $(Q, \tau')^{+A_Q}$, which is the transitive closure of (Q, τ'). As all edges of (Q, τ') appear in $(P, \omega')^{+A_P}$ (recall: it's the same labeling with deleted edges), and $(P, \omega')^{+A_P}$ is transitively closed, we get that all edges of $(Q, \tau')^{+A_Q}$ appear in $(P, \omega')^{+A_P}$. In other words, $S_<(P, \omega')$ must contain all of $S_<(Q, \tau')$. Thus we have that (3)⇒(2).

□

Note (and you can see this clearly by the previous proof) that redundant edges are simply a visual representation of the less-than sets. Thus it is clear how they would be logically equivalent. They have very different benefits however. The method of adding redundant edges and deleting them is far more intuitive, allowing one to more easily see the connection from a simple Hasse diagram. It also has the benefit of not particularly caring what the actual labeling of either poset is. On the other hand, less-than sets are certainly more difficult to visualize and rely on the proper labelings, but mathematically they are far easier to handle rigorously.

Chapter 5

Combining Posets

In this section we provide tools to explain positivity relationships between posets by means of analyzing components of them as opposed to their entirety. This is particularly useful for relationships between posets with many elements, as it is generally much easier to analyze posets of smaller size.

5.1 Sums and Unions

Definition 5.1.1. For posets P and Q, the *ordinal sum*, denoted $P \oplus Q$, is the poset whose order is defined such that for elements $u, v \in P \cup Q$, $u \leq v$ if and only if

- $u, v \in P$ and $u \leq_P v$, or
- $u, v \in Q$ and $u \leq_Q v$, or
- $u \in P$ and $v \in Q$.

For labeled posets (P, ω) and (Q, τ), there are two notions of ordinal sum. In the *weak* (resp. *strict*) *ordinal sum*, denoted $(P, \omega) \oplus^{|} (Q, \tau)$ (resp. $(P, \omega) \oplus^{||} (Q, \tau)$) the

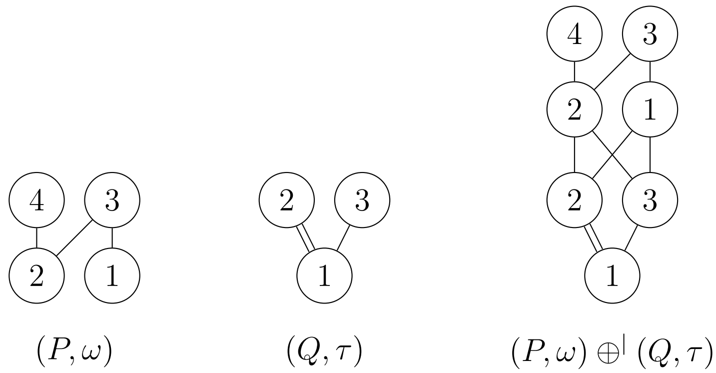


Figure 5.1: Two posets (P, ω) and (Q, τ), and their weak ordinal sum $(P, \omega) \oplus^{|} (Q, \tau)$.

edges from the maximal elements of P to the minimal elements of Q are weak (resp. strict) edges.

For an example of an ordinal sum consider Figure 5.1.

Lemma 5.1.2. *Suppose* $(P_1, \omega_1) \leq_F (Q_1, \tau_1)$ *and* $(P_2, \omega_2) \leq_F (Q_2, \tau_2)$*. Then* $(P_1 \oplus^{|} P_2) \leq_F (Q_1 \oplus^{|} Q_2)$*.*

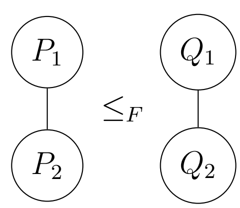


Figure 5.2: Structured of labeled posets as described in Lemma 5.1.2. It is understood that the connection of each maximal element in P_2 (resp. Q_2) to each minimal element in P_1 (resp. Q_1) is represented by the single weak edge between posets.

Proof. To obtain a labeling for $P_1 \oplus^{|} P_2$, let it inherit the labelings from P_1 and P_2 with the modification that the labels for P_2 are all increased by $|P_1|$ and call this new labeling ω. Construct the labeling for $Q_1 \oplus^{|} Q_2$ similarly and call it τ. Both ω and

τ are valid labelings, as all of the labels for P_2 (resp. Q_2) are greater than the labels of P_1 (resp. Q_1), and all of the connections between the two posets are weak. In the interest of brevity of notation, we will omit labelings when referencing the ordinal sums for the remainder of this proof.

By construction of $P_1 \oplus^{|} P_2$, a linear extension of $P_1 \oplus^{|} P_2$ takes the form $(a_1, ..., a_n, b_1, ..., b_m)$ where $(a_1, ..., a_n)$ is a linear extension of P_1 and $(b_1 - |P_1|, ..., b_m - |P_1|)$ a linear extension of P_2. Since $a_n < b_1$ always, the descents of linear extensions of $P_1 \oplus^{|} P_2$ are uniquely constructed from the descents of $(a_1, ..., a_n)$ and $(b_1, ..., b_m)$. A linear extension of $Q_1 \oplus^{|} Q_2$ is determined similarly.

We know that there exists an injective map from the linear extensions of P_1 (resp. P_2) to the linear extensions of Q_1 (resp. Q_2) such that the pattern of descents for a linear extension and its image are the same.

We have then that there must exist an injective mapping g from the linear extensions of $P_1 \oplus^{|} P_2$ to the linear extensions of $Q_1 \oplus^{|} Q_2$ such that the descents of linear extension y and $g(y)$ are the same. This is because since $a_n < b_1$, the descents of linear extensions of $P_1 \oplus^{|} P_2$ or $Q_1 \oplus^{|} Q_2$ are uniquely constructed from the descents of the two linear extensions of their constituent posets.

Since the descent sets determine the F-expansion of $K_{(P,\omega)}$ and $K_{(Q,\tau)}$ by Theorem 2.4.9, we conclude that $(P_1 \oplus^{|} P_2) \leq_F (Q_1 \oplus^{|} Q_2)$. □

Corollary 5.1.3. *Suppose* $(P_1, \omega_1) \leq_F (Q_1, \tau_1)$ *and* $(P_2, \omega_2) \leq_F (Q_2, \tau_2)$. *Then* $(P_1 \oplus^{||} P_2) \leq_F (Q_1 \oplus^{||} Q_2)$.

Proof. Apply Proposition 2.5.2 and then Lemma 5.1.2. □

Definition 5.1.4. We let the *disjoint union* of posets (P, ω) and (Q, τ), denoted $(P + Q, \omega + \tau)$, be the set $P \cup Q$ with the relation defined by $x \leq_{P+Q}$ if $x, y \in P$ and $x \leq_P y$, or if $x, y \in Q$ and $x \leq_Q y$, where the labeling $\omega + \tau$ is defined by $(\omega + \tau)|_P(x) = \omega(x)$ and $(\omega + \tau)|_Q(x) = \tau(x) + |P|$.

We require the following proposition from [MW].

Proposition 5.1.5. *[MW] If (P, ω) and (Q, τ) are labeled posets, then $K_{(P+Q,\omega+\tau)}(x) = K_{(P,\omega)}(x)K_{(Q,\tau)}(x)$.*

Proposition 5.1.6. *If $(P_1, \omega_1) \leq_F (Q_1, \tau_1)$ and $(P_2, \omega_2) \leq_F (Q_2, \tau_2)$, then $(P_1 + P_2, \omega_1 + \omega_2) \leq_F (Q_1 + Q_2, \tau_1 + \tau_2)$.*

Proof. We have that $K_{(Q_1,\tau_1)} - K_{(P_1,\omega_1)} \geq_F 0$. Therefore we know that there is an F-positive quasisymmetric function f such that $K_{(Q_1,\tau_1)} = K_{(P_1,\omega_1)} + f$. Similarly, there is an F-positive g such that $K_{(Q_2,\tau_2)} = K_{(P_2,\omega_2)} + g$. By Proposition 5.1.5 we have that $K_{(P_1+P_2,\omega_1+\omega_2)} = K_{(P_1,\omega_1)}K_{(P_2,\omega_2)}$ and $K_{(Q_1+Q_2,\tau_1+\tau_2)} = K_{(Q_1,\tau_1)}(x)K_{(Q_2,\tau_2)}$. Therefore

$$\begin{aligned}
K_{(Q_1+Q_2,\tau_1+\tau_2)} - K_{(P_1+P_2,\omega_1+\omega_2)} &= \\
&= K_{(Q_1,\tau_1)}(x)K_{(Q_2,\tau_2)} - K_{(P_1,\omega_1)}K_{(P_2,\omega_2)} \\
&= (K_{(P_1,\omega_1)} + f)(K_{(P_2,\omega_2)} + g) - K_{(P_1,\omega_1)}K_{(P_2,\omega_2)} \\
&= K_{(P_1,\omega_1)}f + K_{(P_2,\omega_2)}g + fg
\end{aligned}$$

As the product and sum of F-positive quasisymmetric functions is also F-positive, we conclude that $K_{(Q_1+Q_2,\tau_1+\tau_2)} - K_{(P_1+P_2,\omega_1+\omega_2)}$ is F-positive. □

5.2 The Ur-Operation

We will generalize some of our results to different combinations of posets using an operation that allows us to "compose" posets.

Definition 5.2.1. [BHK17] For a poset $\mathcal{P} = \{x_1, ..., x_n\}$ and a sequence of posets $(P_1, P_2, ..., P_n)$, we define the *Ur-operation* on $\mathcal{P}$ of $(P_1, P_2, ..., P_n)$ as the poset $\mathcal{P}[x_i \rightarrow$

$P_i]_{i=1}^n$ on $\cup_{i=1}^n P_i$ with the following order relation:

$$\text{For } p \in P_j,\ q \in P_k,\ p \leq q \text{ when } \begin{cases} p \leq_{P_j} q & j = k \\ x_j \leq_{\mathcal{P}} x_k & j \neq k \end{cases}, \tag{5.1}$$

We will refer to P_i as *composite* posets and $\mathcal{P}$ as the *parent* poset.

Since if $|P| = n$ there will always be n maps from elements to posets, we will notate the Ur-operation in this section as $\mathcal{P}[x_i \to P_i]$ rather than $\mathcal{P}[x_i \to P_i]_{i=1}^n$.

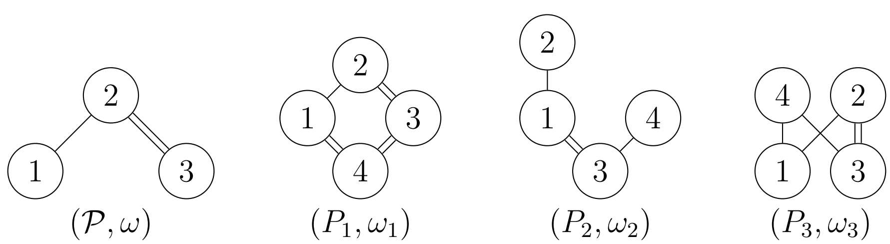
Note that this is a generalization of the *ordinal product* operation as defined in section 3.2 of [Sta12].

Definition 5.2.2. For a labeled poset $(\mathcal{P}, \omega)$ with elements $\{x_1, ..., x_n\}$ and a sequence of labeled posets $\{(P_1, \omega_1), (P_2, \omega_2), ..., (P_n, \omega_n)\}$, we define the *inherited labeling* Ω on $\mathcal{P}[x_i \to P_i]$ as

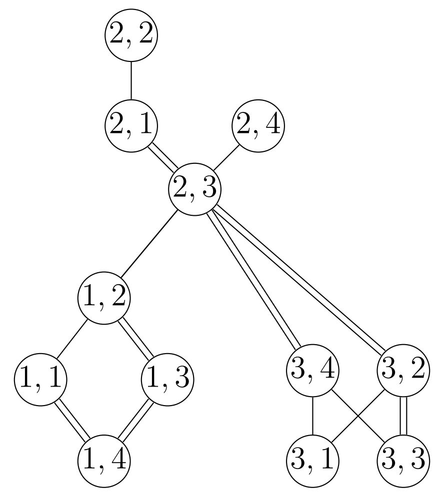
$$\text{For } p \in P_l,\ \Omega(p) = (\omega(x_l), \omega_l(p)),$$

where the total order is imposed lexicographically.

For example, take the following posets:



If we are to apply the Ur-operation and give the resulting poset its inherited labeling, we get the following:



$(\mathcal{P}[x_i \to P_i], \Omega)$

Note that the inherited label preserves the strict and weak edges not only within the each composite poset P_i, but also between composite posets. Whenever labeling a Ur-operation poset we will use this inherited labeling; for example $\mathcal{L}(\mathcal{P}[x_i \to P_i], \Omega)$ and $\mathcal{L}(\mathcal{Q}[y_i \to Q_i], \Upsilon)$ in the following section have Ω and Υ as the inherited labeling.

Remark 5.2.3. The less-than set for $\mathcal{P}[x_i \to P_i]$ can be read off from (5.1). Moreover, two elements associated with labels (a_1, b_1) and (a_2, b_2) have the relation $(a_1, b_1) < a_2, b_2)$ if and only if $\omega^{-1}(a_1) <_{\mathcal{P}} \omega^{-1}(a_2)$, or $\omega^{-1}(a_1) = \omega^{-1}(a_2)$ (both in P_i) and $\omega_i^{-1}(b_1) <_{P_i} \omega_i^{-1}(b_2)$. This follows *directly* from the construction in (5.1).

Theorem 5.2.4. *If $\mathcal{L}(\mathcal{P}, \omega) \subseteq \mathcal{L}(\mathcal{Q}, \tau)$ and $\mathcal{L}(P_i, \omega_i) \subseteq \mathcal{L}(Q_i, \tau_i)$ for all i such that $1 \leq i \leq |\mathcal{P}|$, and each element x_j and y_j corresponds to the element of $\mathcal{P}$ and $\mathcal{Q}$ respectively such that $\omega(x_j) = \tau(y_j) = j$, then $\mathcal{L}(\mathcal{P}[x_i \to P_i], \Omega) \subseteq \mathcal{L}(\mathcal{Q}[y_i \to Q_i], \Upsilon)$.*

It is not so difficult to see why this might be true, especially when looking at the composed Hasse diagrams. Consider what changes would be made there to the connections in the parent and composite posets, and how those would connect to each other. For an example see Figure 5.3. The formal proof below will of course use the more mathematically-friendly less-than set.

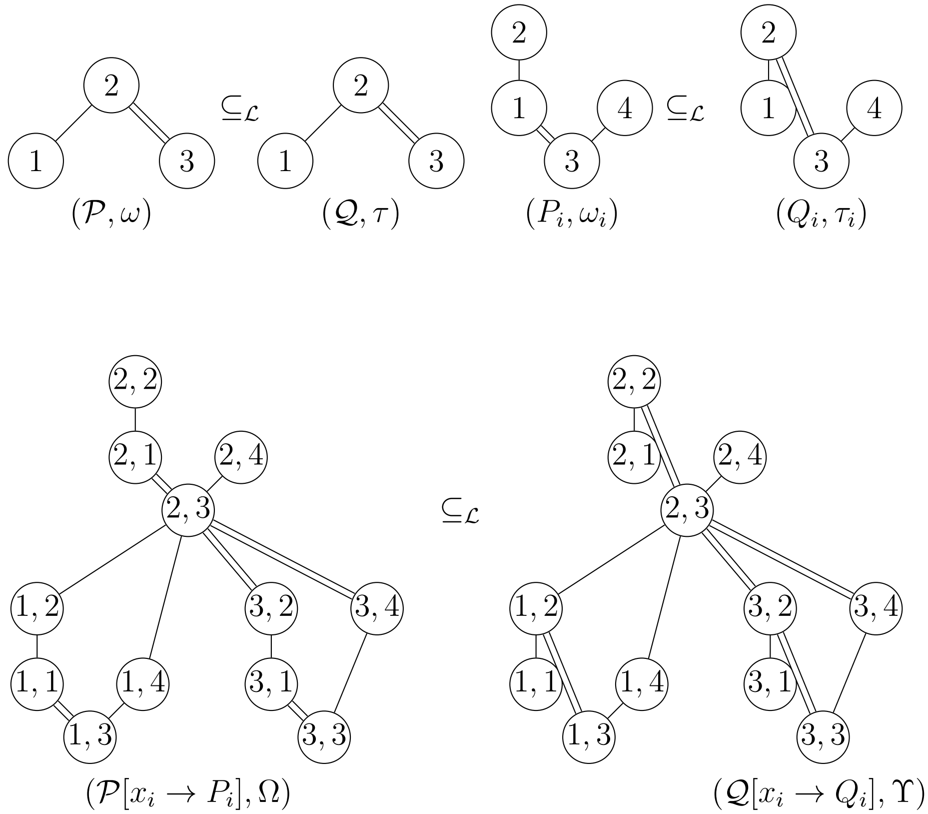


Figure 5.3: Four posets $(\mathcal{P}, \omega)$, $(\mathcal{Q}, \tau)$, (P_i, ω_i), and (Q_i, τ_i), with relations $\mathcal{L}(\mathcal{P}, \omega) \subseteq \mathcal{L}(Q, \tau)$ and $\mathcal{L}(P_i, \omega_i) \subseteq \mathcal{L}(Q_i, \tau_i)$. Also pictured is the resultant relation $\mathcal{L}(\mathcal{P}[x_i \to P_i], \Omega) \subseteq \mathcal{L}(\mathcal{Q}[x_i \to Q_i], \Upsilon)$.

Proof of Theorem 5.2.4. We will proceed by considering the less-than sets for $\mathcal{P}[x_i \to P_i]$ and $\mathcal{Q}[y_i \to Q_i]$, and then apply Lemma 4.3.5. We have by the definition of the Ur-operation that $a <_{\mathcal{P}[x_i \to P_i]} b$ if and only if either $a, b \in P_k$ and $a \leq_{P_k} b$, or $a \in P_i$, $b \in P_j$ and $x_i <_{\mathcal{P}} x_j$. The same applies to $\mathcal{Q}[y_i \to Q_i]$. Take a less-than rule $(\alpha, \beta) \in S_<(\mathcal{Q}[y_i \to Q_i])$. Note that each of α and β is in fact a tuple, which we will reference accordingly as (α_1, α_2) and (β_1, β_2). We will seek to show that $(\alpha, \beta) \in S_<(\mathcal{P}[x_i \to P_i])$.

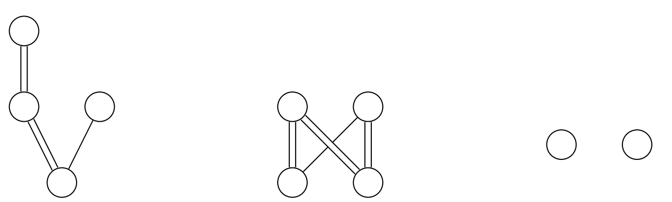
There are two cases we will need to consider: that the poset elements that are assigned α and β originate from the same composite poset, or they do not.

First assume that $\Upsilon^{-1}(\alpha), \Upsilon^{-1}(\beta) \in Q_k$ for some k. This means that $\alpha_1 = \beta_1$. Since $\mathcal{L}(P_k, \omega_k) \subseteq \mathcal{L}(Q_k, \tau_k)$ and $(\alpha_2, \beta_2) \in S_<(Q_k, \tau_k)$, we have by Lemma 4.3.5 that $(\alpha_2, \beta_2) \in S_<(P_k, \omega_k)$. Thus $\omega_k^{-1}(\alpha) <_{P_k} \omega_k^{-1}(\beta)$, and therefore $\Omega^{-1}(\alpha) <_{\mathcal{P}[x_i \to P_i]} \Omega^{-1}(\beta)$. Thus $(\alpha, \beta) \in S_<(\mathcal{P}[x_i \to P_i])$.

Now assume that $\Upsilon^{-1}(\alpha) \in Q_k$ and $\Upsilon^{-1}(\beta) \in Q_l$ for $k \neq l$. We know that $y_k \leq_{\mathcal{Q}} y_l$. This means that $x_k \leq_{\mathcal{P}} x_l$ by Lemma 4.3.5 and our assumption that $\mathcal{L}(\mathcal{P}, \omega) \subseteq \mathcal{L}(\mathcal{Q}, \tau)$. We also have that $\Omega^{-1}(\alpha) \in P_k$ and $\Omega^{-1}(\beta) \in P_l$. This means that $\Omega^{-1}(\alpha) \leq_{\mathcal{P}[x_i \to P_i]} \Omega^{-1}(\beta)$, and therefore $(\alpha, \beta) \in S_<(\mathcal{P}[x_i \to P_i])$.

In either case, since (α, β) was arbitrary, we have that $S_<(\mathcal{P}[x_i \to P_i]) \supseteq S_<(\mathcal{Q}[y_i \to Q_i])$, and by Lemma 4.3.5 we have that $\mathcal{L}(\mathcal{P}[x_i \to P_i]) \subseteq \mathcal{L}(\mathcal{Q}[y_i \to Q_i])$. □

This concludes our discussion on Ur-operation posets that have linear extension containment. We next consider the question of when the Ur-operation preserves F-positivity. This is not always the case; for example the posets



$(\mathcal{Q}, \tau)$ $(\mathcal{P}, \omega)$ (P_i, ω_i) for all i

are such that $(\mathcal{P}, \omega) \leq_F (\mathcal{Q}, \tau)$, however it is not the case that $(\mathcal{P}[x_i \to P_i], \Omega) \leq_F (\mathcal{Q}[y_i \to P_i], \Upsilon)$, where Ω and Υ denote the inherited labelings. It must be noted however that it is the case that the F-support of $(\mathcal{Q}[y_i \to P_i], \Upsilon)$ contains the F-support of $(\mathcal{P}[x_i \to P_i], \Omega)$. This actually works for all two-element posets P_i and four-element posets $\mathcal{P}$ and Q, and for at least half of the three-element posets P_i and four-element posets $\mathcal{P}$ and $\mathcal{Q}$.

There are certain restrictions we can place on the elements of the Ur-operation that are weaker than those in Theorem 5.2.4. Additionally, for ease of notation we will identify the elements of T with their labels, much as we had done in Section 4.3.

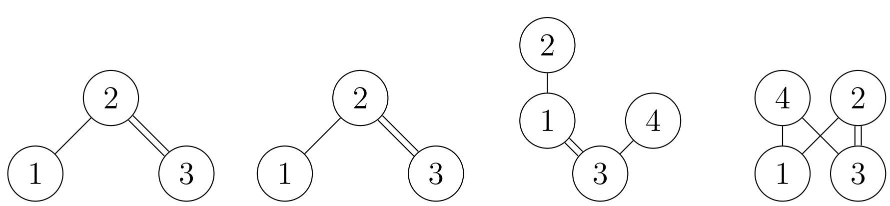
The following theorem is the second main result of this section, and serves to generalize several of the relations seen in the first section of this chapter.

Theorem 5.2.5. *Take labeled posets $(\mathcal{P}, \omega)$, $(\mathcal{Q}, \tau)$, and ordered sets of labeled posets $\{(P_1, \omega_1), ..., (P_n, \omega_n)\}$ and $\{(Q_1, \tau_1), ..., (Q_n, \tau_n)\}$. If $\mathcal{L}(\mathcal{P}, \omega) \subseteq \mathcal{L}(\mathcal{Q}, \tau)$ and $(P_i, \omega_i) \leq_F (Q_i, \tau_i)$ for all $i = 1, ..., n$, then $(\mathcal{P}[x_i \to P_i], \Omega) \leq_F (\mathcal{Q}[y_i \to Q_i], \Upsilon)$.*

We will construct a map $\Phi\colon \mathcal{L}(\mathcal{P}[x_i \to P_i], \Omega) \to \mathcal{L}(\mathcal{Q}[y_i \to Q_i], \Upsilon)$. In more readable terms, Φ alters the tuples in a given linear extension of $(\mathcal{P}[x_i \to P_i], \Omega)$ such that the first element of the tuple—the one originating in ω—is unchanged, while the second element of the tuple—the one originating in ω_i—is mapped to the corresponding label in τ_i.

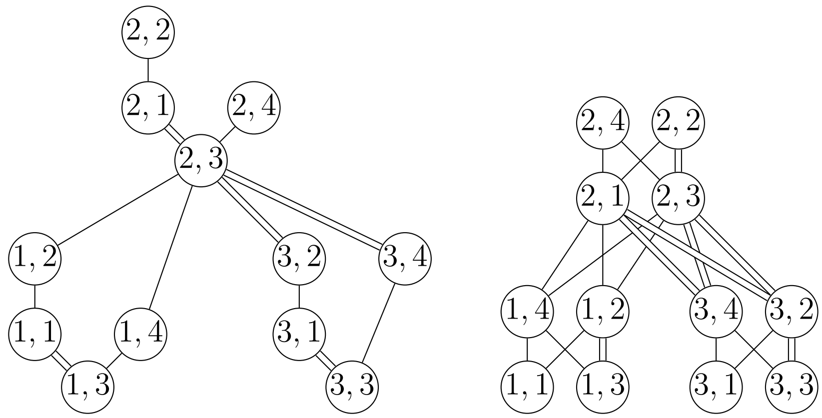
This process (especially when mathematically defined as you will soon see) can be very difficult to visualize. The following is an example meant to assist in this by showing a specific Ur-Operation construction and how Φ acts upon one of its linear extensions.

Take the following labeled posets:



$(\mathcal{P}, \omega)$ $(\mathcal{Q}, \tau)$ (P_i, ω_i) (Q_i, τ_i)

The next step is to construct the posets $(\mathcal{P}[x_i \to P_i], \Omega)$ and $(\mathcal{Q}[x_i \to Q_i], \Upsilon)$, which we do below with the inherited labelings Ω and Υ.



$(T[x_i \to P_i], \Omega)$ $(\mathcal{Q}[x_i \to Q_i], \Upsilon)$

Now we can take a linear extension in $(\mathcal{P}[x_i \to P_i], \Omega)$, and show how Φ will transform it in to a linear extension of $(\mathcal{Q}[x_i \to Q_i], \Upsilon)$.

One linear extension of $(\mathcal{P}[x_i \to P_i], \Omega)$ is

$$\pi = (3,3),(3,4),(1,3),(1,1),(1,2),(1,4),(3,1),(3,2),(2,3),(2,1),(2,4),(2,2).$$

The first step of course is to determine the sub-sequences associated with each composite poset. This is done by grouping by the first index in the tuple-labels, shown by the colors. We get three sub-sequences (since $|T| = 3$), which are $(1,3),(1,1),(1,2),(1,4)$ for P_1, $(2,3),(2,1),(2,4),(2,2)$ for P_2, and $(3,3),(3,4),(3,1),(3,2)$ for P_3.

The next step is to determine how to match up the second linear extensions. It turns out we can map $3124 \to 3124$ for $P_1 \to Q_1$, $3142 \to 3124$ for $P_2 \to Q_2$, and $3412 \to 1324$ for $P_3 \to Q_3$ to preserve the descents of the linear extensions.

Now we can construct $\Phi(\pi)$. If we swap the corresponding parts of π, we get a new linear extensions,

$$\Phi(\pi) = (3,1),(3,3),(1,3),(1,1),(1,2),(1,4),(3,2),(3,4),(2,3),(2,1),(2,4),(2,2).$$

One can check that this is in fact a linear extension for $(\mathcal{Q}[x_i \to Q_i], \Upsilon)$, as we expected. One can also check that the descents of π are the same as the descents of $\Phi(\pi)$. This concludes our example; we will now construct our injection Φ more rigorously.

Since $(P_i, \omega_i) \leq_F (Q_i, \tau_i)$, there exists an injective function $\zeta^i\colon \mathcal{L}(P_i, \omega_i) \to \mathcal{L}(Q_i, \tau_i)$ such that $Des(\beta) = Des(\zeta^i(\beta))$ for each i. Now let

$$\pi = (\pi_1, ..., \pi_k) = ((\alpha_1, \beta_1), ..., (\alpha_k, \beta_k)) \in \mathcal{L}(\mathcal{P}[x_i \to P_i], \Omega).$$

If we fix an α_i for this π, we know that the the corresponding β_j form a linear extension for P_{α_i}. We will refer to this linear extension as

$$\pi_{\alpha_i} = (\alpha_i, \beta_1), (\alpha_i, \beta_2), (\alpha_i, \beta_3), ..., (\alpha_i, \beta_{|P_i|}).$$

Now we know that the second elements of each of these pairs forms a linear extension β of (P_i, ω_i) to which we can apply ζ^i. Let the jth element of $\zeta^i(\beta)$ be denoted by a subscript: $\zeta^i(\beta)_j$. We construct Φ as the function that replaces every such subsequence $(\alpha_i, \beta_1), (\alpha_i, \beta_2), ..., (\alpha_i, \beta_{|P_i|})$ in π by $(\alpha_i, \zeta^i(\beta)_1), (\alpha_i, \zeta^i(\beta)_2), ..., (\alpha_i, \zeta^i(\beta)_{|P_i|})$.

This is what we did in our example: we keep all of the first terms in the tuples of the sequence the same, and map each of the second terms for a linear extension of (P_i, ω_i) to their counterpart in the corresponding linear extension of (Q_i, τ_i). To prove Theorem 5.2.5, we will have to show that this function injects to the linear extensions of $\mathcal{L}(\mathcal{Q}[y_i \to Q_i], \Upsilon)$ in such a way that it preserves the descents.

Lemma 5.2.6. *If* $\pi \in \mathcal{L}(\mathcal{P}[x_i \to P_i], \Omega)$*, then* $\Phi(\pi) \in \mathcal{L}(\mathcal{Q}[y_i \to Q_i], \Upsilon)$.

Proof. It is clear that $\Phi(\pi)$ maps to the set of valid tuple labelings of $(\mathcal{Q}[y_i \to Q_i], \Upsilon)$, thus we simply need to show that $\Phi(\pi)$ is a linear extension of $(\mathcal{Q}[y_i \to Q_i], \Upsilon)$, i.e. it does not violate a less-than rule for $(\mathcal{Q}[y_i \to Q_i], \Upsilon)$.

Take two tuple elements of $\Phi(\pi)$: an $(\alpha_i, \zeta^i(\beta)_\ell)$ that comes before $(\alpha_j, \zeta^j(\beta)_k)$. We claim that $((\alpha_j, \zeta^j(\beta)_k), (\alpha_i, \zeta^i(\beta)_\ell)) \notin S_<(\mathcal{Q}[y_i \to Q_i], \Upsilon)$, i.e. there is not a rule in the less-than set of $(\mathcal{Q}[y_i \to Q_i], \Upsilon)$ that prohibits $(\alpha_i, \zeta^i(\beta)_\ell)$ from appearing before $(\alpha_j, \zeta^j(\beta)_k)$.

First say that the potential violation comes from 2 posets being mapped to elements of the same label in $(\mathcal{P}, \omega)$ and $(\mathcal{Q}, \tau)$ (i.e. they are (P_i, ω_i) and (Q_i, τ_i), the same i). In terms of our labels, this means that $\alpha_i = \alpha_j$, and therefore $\zeta^i = \zeta^j$. This also means that β_ℓ and β_k come from the same linear extension β of P_i, which of course implies that $\ell < k$. Since $\zeta^i(\beta)$ is a linear extension of Q_i, this means that $\zeta^i(\beta)_k \not\leq_{P_{\alpha_i}} \zeta^i(\beta)_\ell$. By the definition of the Ur-operation this means that $(\alpha_j, \zeta^j(\beta)_k) \not\leq_{(\mathcal{Q}[y_i \to Q_i], \Upsilon)} (\alpha_i, \zeta^i(\beta)_\ell)$, and we have our claim in this case.

Now assume that the potential violation comes from different composite posets. In terms of our labels, this means that $\alpha_i \neq \alpha_j$. We know however that $\omega^{-1}(\alpha_i) \ngeq \omega^{-1}(\alpha_j)$. Since $\mathcal{L}(\mathcal{P}, \omega) \subseteq \mathcal{L}(\mathcal{Q}, \tau)$, we know that all relations of labels in $(\mathcal{Q}, \tau)$ are preserved in $(\mathcal{P}, \omega)$. First this guarantees that α_i and α_j are in fact labels for $(\mathcal{Q}, \tau)$, but it also means that $\tau^{-1}(\alpha_i) \ngeq \tau^{-1}(\alpha_j)$. Thus we know that $((\alpha_j, \zeta^j(\beta)_k), (\alpha_i,$ $zeta^i(\beta)_\ell) \notin S_<(\mathcal{Q}[y_i \to Q_i], \Upsilon)$.

Thus we have that in either case, $((\alpha_j, \zeta^j(\beta)_k), (\alpha_i, \zeta^i(\beta)_\ell)) \notin S_<(\mathcal{Q}[y_i \to Q_i], \Upsilon)$. Since $(\alpha_i, \phi_{\pi_{\alpha_i}}(\beta_i))$ and $(\alpha_j, \phi_{\pi_{\alpha_j}}(\beta_j))$ were arbitrary pairs of elements of $\Phi(\pi)$, we know that there exists no violations of $S_<(\mathcal{Q}[y_i \to Q_i], \Upsilon)$ in $\Phi(\pi)$. Since there exist no violations and all labels are represented, we have that $\Phi(\pi)$ is a linear extension of $(\mathcal{Q}[y_i \to Q_i], \Upsilon)$. □

Lemma 5.2.7. Φ *is injective.*

In short the injectivity comes from the fact that Φ does nothing to the first elements and ζ^i injects on the second elements. It makes sense that these two operations combined would result in an injection.

Proof of Lemma 5.2.7. Say that $\Phi(\pi) = \Phi(\gamma)$. Say for contradiction that π and γ differ at some index r, call the tuples there $(a(\pi)_r, b(\pi)_r)$ and $(a(\gamma)_r, b(\gamma)_r)$. Since the first element of each tuple remains unchanged by Φ, we know that the first elements $a(\pi)_r$ and $a(\gamma)_r$ must be the same. Since this is true for all tuples, we know that $(a(\pi)_r, b(\pi)_r)$ and $(a(\gamma)_r, b(\gamma)_r)$ must both be the kth tuple with first element $a(\pi)_r = a(\gamma)_r$. This means that under Φ, $b(\pi)_r$ and $b(\gamma)_r$ both get mapped to $\zeta^a(\beta)_k$ for some β. Since ζ^i is injective for all i, we get that $b(\pi)_r = b(\gamma)_r$. Thus $(a(\pi)_r, b(\pi)_r) = (a(\gamma)_r, b(\gamma)_r)$, and we know that π and γ cannot differ at any index. □

We have now that Φ is an injection from $\mathcal{L}(\mathcal{P}[x_i \to P_i], \Omega)$ to $\mathcal{L}(\mathcal{Q}[y_i \to Q_i], \Upsilon)$. The final piece is to show that descents are preserved.

Lemma 5.2.8. *If $\pi \in \mathcal{L}(\mathcal{P}[x_i \to P_i], \Omega)$, then $Des(\pi) = Des(\Phi(\pi))$.*

Proof. Take two adjacent elements $(..., (\alpha_i, \beta_i), (\alpha_j, \beta_j), ...) = \pi$. For all j, let α^j denote the linear extension associated with α_j, and suppose $\beta_j = (\lambda^j)_{k_j}$. We seek to show that $(\alpha_i, \beta_i) \leq_{lex} (\alpha_{i+1}, \beta_{i+1})$ if and only if $(\alpha_i, \zeta^i(\lambda^i)_{k_i}) \leq_{lex} (\alpha_{i+1}, \zeta^{i+1}(\lambda^{i+1})_{k_{i+1}})$.

Say first that $\alpha_i \neq \alpha_{i+1}$. Since the first element of the tuple is unchanged by Φ and lexicographic order relies on the first element first, we know that the lexicographic order of the tuples will hold.

Now say that $\alpha_i = \alpha_{i+1}$. This implies three things: first that $\zeta^i = \zeta^{i+1}$, that $\lambda^i = \lambda^{i+1}$, and that $k_{i+1} = k_i + 1$. Since ζ^i preserves descents, we know that $\beta_i < \beta_{i+1}$ if and only if $\zeta^i(\lambda^i)_{k_i} < \zeta^i(\lambda^i)_{k_i+1} = \zeta^{i+1}(\lambda^{i+1})_{k_{i+1}}$. Since the ordering is lexicographic and $\alpha_i = \alpha_{i+1}$, this means that $(\alpha_i, \beta_i) \leq_{lex} (\alpha_{i+1}, \beta_{i+1})$ if and only if $(\alpha_i, \zeta^i(\lambda^i)_{k_i}) \leq_{lex} (\alpha_{i+1}, \zeta^{i+1}(\lambda^{i+1})_{k_{i+1}})$.

Thus in both cases we have our claim, and we have that Φ preserves descents.

□

We are finally ready to prove Theorem 5.2.5, which follows immediately:

Proof of Theorem 5.2.5. This follows directly from Lemmas 5.2.6, 5.2.7, and5.2.8. □

Note that in this case, there is no reason why $(T, \omega) \neq (\mathcal{Q}, \tau)$, or why $(P_i, \omega_i) \neq (Q_i, \tau_i)$. For this reason, Theorem 5.2.5 generalizes several of our previous results, namely Lemma 5.1.2, Corollary 5.1.3, and Proposition 5.1.6. In other words, the results for ordinal sum *and* disjoint union of labeled posets are both special cases of Theorem 5.2.5.

Chapter 6

Deletion and Reversal

This section explores the relationships that arise from the addition (or subtraction) of valid edges from posets. When there is no relationship between two elements x, y of (P, ω), every (P, ω)-partition either satisfies $f(x) \leq f(y)$ or $f(x) > f(y)$. Thus we can write the generating function as a sum of the generating function of two other posets, one where $x \leq_P y$ and one where $x >_P y$. Note the strictness of these inequalities could just as easily be switched, where $f(x) \geq f(y)$ or $f(x) < f(y)$. For an edge that is $x \leq_P y$ (resp. $x >_P y$), we call its reversal the edge that denotes $x >_P y$ (resp. $x \leq_P y$), and vice versa. For notational purposes, in the following section we will label a poset (P, ω) such that $(P, \omega)^{d(e)}$ is the poset that results from deleting an edge e, and $(P, \omega)^{r(e)}$ is the poset that results from that same edge's reversal. We have then the relationship

$$(P, \omega) + (P, \omega)^{r(e)} = (P, \omega)^{d(e)} \text{ or, equivalently, } (P, \omega) = (P, \omega)^{d(e)} - (P, \omega)^{r(e)}.$$

For an example see Figure 6.1.

Proposition 6.0.1. *If deleting edges e_1 and e_2 from posets (P, ω) and (Q, τ) respectively are such that $(P, \omega)^{d(e_1)} \geq_F (Q, \tau)^{d(e_2)}$ and the reversal of that edge is such that $(P, \omega)^{r(e_1)} \leq_F (Q, \tau)^{r(e_2)}$, then $(P, \omega) \geq_F (Q, \tau)$.*

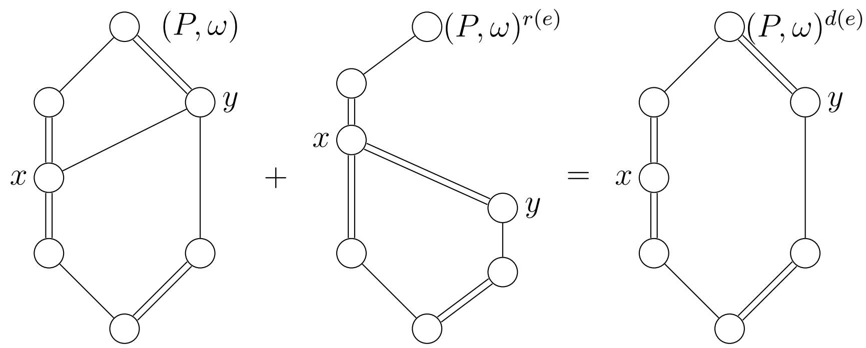


Figure 6.1: An example of three posets (P,ω), $(P,\omega)^{d(e)}$, $(P,\omega)^{r(e)}$ and their relation to one another.

Proof. We have that

$$\begin{aligned}(P,\omega)-(Q,\tau) &= ((P,\omega)^{d(e_1)}-(P,\omega)^{r(e_1)})-((Q,\tau)^{d(e_2)}-(Q,\tau)^{r(e_2)})\\ &= ((P,\omega)^{d(e_1)}-(Q,\tau)^{d(e_2)})+((Q,\tau)^{r(e_2)}-(P,\omega)^{r(e_1)}).\end{aligned}$$

Since we know that $(P,\omega)^{d(e_1)}-(Q,\tau)^{d(e_2)} \geq_F 0$ and $(Q,\tau)^{r(e_2)}-(P,\omega)^{r(e_1)} \geq_F 0$, we know that there sum is also F-positive and therefore $(P,\omega)-(Q,\tau) \geq_F 0$. □

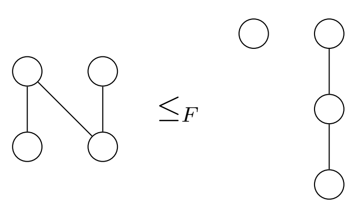
Using very similar methods we get the following results:

Proposition 6.0.2. *Take labeled posets (P,ω) and (Q,τ).*

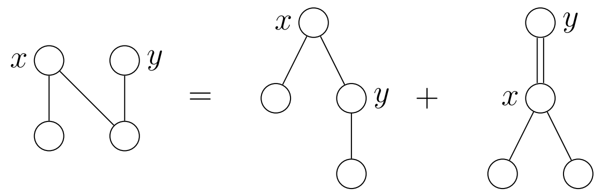
(a) If $(P,\omega) \leq_F (Q,\tau)$ and $(P,\omega)^{d(e_1)} \geq (Q,\tau)^{d(e_2)}$, then we have that $(P,\omega)^{r(e_1)} \geq_F (Q,\tau)^{r(e_2)}$.

(b) If $(P,\omega) \leq_F (Q,\tau)$ and $(P,\omega)^{r(e_1)} \leq_F (Q,\tau)^{r(e_2)}$, then $(P,\omega)^{d(e_1)} \leq_F (Q,\tau)^{d(e_2)}$.

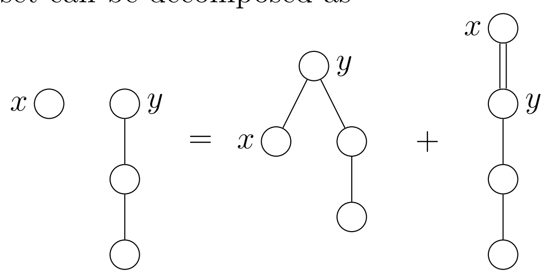
We use these methods to explain inequalities such as



Where the lesser poset can be decomposed as



and the greater poset can be decomposed as



We know by Theorem 4.4.1 that the second elements of these sums give us the relation that we desire by Proposition 6.0.2 (b). This can be accomplished by adding a weak redundant edge from lowest element in the chain to the third lowest element in the chain, then deleting the lowest edge.

Chapter 7

Cases with Necessary and Sufficient Conditions

In this chapter we look for classes of posets for which our conditions are both necessary and sufficient to explain all relations. In this chapter we will first concern ourselves with posets of all weak (or equivalently all strict by the star involution) edges. These are called *naturally labeled posets*, because the weak version has all of the labels increasing. When we refer to the jump or star-jump sequences of these posets, we are referring to the jump as we defined it under all strict edges. Additionally when referencing to a naturally labeled poset (P, ω), we will forego the ω and simply note P.

7.1 Greene Shape $(k, 1)$

This section is concerned with a naturally labeled poset's Greene shape, defined below. Here we will restrict our search for conditions to posets of a particular Greene shape—namely $(k, 1)$—and find both necessary and sufficient conditions based on the

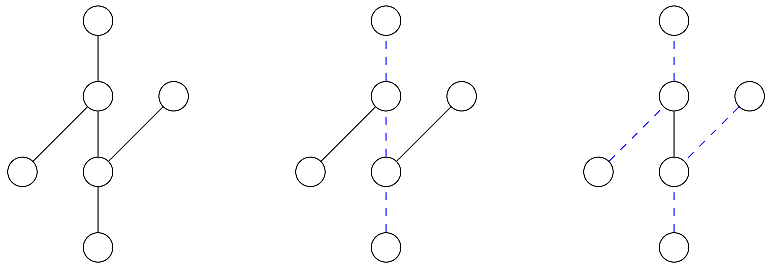


Figure 7.1: A naturally labeled poset and the maximal unions of one and two chains that gives its Greene shape of $(4, 2)$

jump sequences. The study of these shapes in the context of our work was inspired by [IW18].

Definition 7.1.1. [Gre76] For naturally labeled poset P and $k \geq 0$, let c_k represent the maximal cardinality of a union of k chains in P. Then the *Greene shape* of a poset is the sequence $(\lambda_1, \lambda_2, ...,)$ where $\lambda_i = c_i - c_{i-1}$.

The first class we consider is those labeled posets that consist of a single chain, referred to as the *spine*, and a single other element. These are all of the naturally labeled posets of Greene shape $(k, 1)$. There are four possible ways that this element can be connected, which can be seen in the example in Figure 7.2. Note that these chains are of arbitrary length. The extra element p in Type I posets are covered by a non-minimal element of the spine. In Type II posets the p covers a non-maximal element of the spine. In Type III posets p both is covered by and covers elements of the spine. Note that these two connections cannot be to adjacent nodes, or else the connection on the spine would be redundant and the poset would become a single chain. Type IV posets consist of the spine and a single disconnected element. This complete classification of connections allows us to narrow our search for conditions and simplify the representation of the poset.

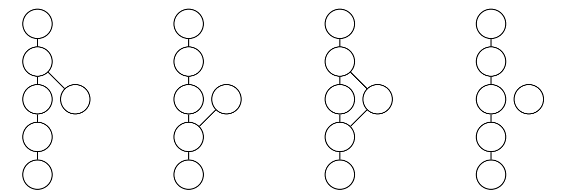


Figure 7.2: Examples of the four different ways an additional node can be connected to a chain of any length

We only wish to compare posets of the same size, and since there is only one element not in the spine, we know that the number of elements in the spine of two comparable naturally labeled posets of Greene shape $(k, 1)$ are going to be the same.

We can also change how we encode this poset. We will define the *height* of an element q in the spine to be the number of nodes in a saturated chain from q to the minimal element of the spine. It is clear then that for a naturally labeled poset P of shape $(k, 1)$ that the height of the maximal element of the spine will be $k = |P| - 1$. Letting p denote the non-spine element of P, we can therefore encode P in tuples where the first entry is the height of element that p covers (0 if there is none) and the second entry is the height of the spine element that p is covered by ($|P|$ if there is none). For example all of the encodings for all possible such posets of shape $(5, 1)$ can be seen in Figure 7.3. For the different types, the tuples of Type I will be for the form $(0, b)$, Type II of the form $(a, |P|)$, Type III of the form (a, b), and Type IV of the form $(0, |P|)$, where $0 < a < b < |P|$. We will denote the tuple representation of P as $T(P)$.

We can determine the jump and star-jump sequence of each type of poset from these tuples as well. For example the jump for poset P where $T(P) = (0, b)$ will be $(2, 1, 1, ..., 1)$ and the star-jump will be $(1, 1, ..., 1, 2, 1, ..., 1)$, where the 2 in the star-

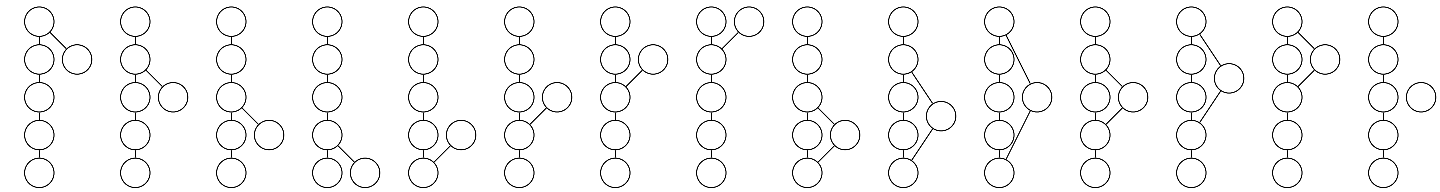


Figure 7.3: All of the tuple encodings for all naturally labeled posets of Greene shape $(5, 1)$.

jump occurs at position $|P| - b + 1$ of the sequence. A poset P where $T(P) = (a, |P|)$ will be formulated similarly with the star-jump and the jump switched. A poset P with $T(P) = (a, b)$ will have jump of $(1, 1, ..., 1, 2, 1, ..., 1)$ where the 2 in the jump occurs at position $a + 1$, and a star-jump formulated the same way as for Type I. Any poset of Type IV will of course have jump and star-jump of $(2, 1, ..., 1)$.

We can also view these tuples as intervals on the real line, which, abusing notation, we will also denote $T(P)$. Note then that if $T(P) = (a, b)$, then $T(P^*) = (|P| - b, |P| - a)$. Thus as intervals $T(P) \subset T(Q)$ if and only if $T(P^*) \subset T(Q^*)$. We can formulate then the following proposition which gives a simple necessary and sufficient condition for whowing F-positivity between posets of Greene shape $(k, 1)$.

Theorem 7.1.2. *Suppose naturally labeled posets P and Q both have Greene shape $(|P| - 1, 1)$. Then $P <_F Q$ if and only if $T(P) \subset T(Q)$.*

Proof. First note that $P =_F Q$ if and only if $T(P) = T(Q)$. We know that if $T(P) = T(Q)$ then clearly P and Q are the same, and therefore $P =_F Q$. If $T(P) \neq T(Q)$, then we have realized that the jump or star-jump of P and Q will not be equal, and therefore $P \neq_F Q$. Since we are assuming inequality, all relations are strict F-inequalities ($<_F$) rather than our usual weak F-inequalities ($\leq_F$). The remainder of this proof will be done in cases, looking at the different posets capable of being constructed, as seen in Figure 7.2.

Case 1—P or Q of Type IV.

It is clear that in the case of any comparison involving poset Q of Type IV and another poset P of type I, II, or III, the Type IV poset Q will always be greater than P by edge deletion (Proposition 4.1.1, and the associated tuple $T(P)$ will always be contained in $T(Q)$ since $T(Q) = (0, |P|)$.

Case 2—P and Q of types I and/or II

First take relations involving P and Q both of Type I. There are two possibilities: either the connection to the spine is the same (in which case the tuples and F-expansion are also equal), or they are different and in one poset the extra element is connected higher than the other. Say that $T(P) \subset T(Q)$, i.e. $T(P) = (0, b_1)$ and $T(Q) = (0, b_2)$ where $b_1 < b_2$. We know P must be F-less than Q (by Theorem 4.4.1), because Q is equal to the addition of a redundant edge from the extra element up to the b_2th element of the chain to P, and subsequent deletion of the original edge in P. Similarly if $T(P) \not\subset T(Q)$, then we know that $T(Q) \subseteq T(P)$ and we know from the process above that $Q \leq_F P$ and therefore $P \not<_F Q$. Thus the proposition holds.

Since Type II posets are simply Type I posets under the star-involution, by symmetry we know that similar rules must apply to relationships between two Type II posets. Thus we know our proposition holds when P and Q are posets of Type II.

In the sub-case of comparing a poset P of Type I to a poset Q of Type II, dominance order on the jump tells us that $P \not<_F Q$, and dominance order on the star-jump would have us believe that $Q \not<_F P$. For the other direction, in this case $T(P) = (0, b_1)$ with $b_1 < |P|$ and $T(Q) = (a_1, |P|)$ with $a_1 > 0$. Clearly $T(P) \not\subset T(Q)$ and $T(Q) \not\subset T(P)$, consistent with our proposition.

Case 3—P or Q (not both) are Type III

Now we must consider relations containing a poset P of Type III. Since applying the star involution on posets of Type II give posets of Type I and applying the star involution on posets of Type III give other posets of Type III, we can simply consider the relations between posets of types I and III. Let Q be a poset of Type I. From the dominance order on the jump sequence, we know that if there is a relation, it must be that $P <_F Q$. This is consistent with our proposition as it is similarly only possible in this case for $T(P) \subset T(Q)$ if P is the poset of Type III and Q is the poset of Type I.

Assume then that $T(P) \subset T(Q)$. This means that the extra element in Q is connected to the spine of Q at least as high as than both of the connections for the extra element to the spine of P. Thus if we delete the lower edge of P and raise the connection on the upper edge by redundant addition and deletion, we can obtain Q. By Theorem 4.4.1 we know that $P <_F Q$.

Now assume that $T(P) \not\subset T(Q)$. This means that if $T(Q) = (0, b_2)$ and $T(P) = (a_1, b_1)$ then $b_2 < b_1$. That would mean however that the star-jump of P is dominant over the star-jump of Q, and therefore $P \not<_F Q$. Thus we know that our proposition holds in the case of comparing a Type III and Type I (and II) poset.

Case 4—P and Q of type III

Finally we will consider relations between posets P and Q both of Type III. Assume that $T(P) \subset T(Q)$. Then there are two cases. First the upper (resp. lower) connection for Q is higher (resp. lower) than the upper (resp. lower) connection for P. Then it is clear that by the addition of two redundant edges to P and deletion of two edges that $P <_F Q$. Secondly, if only one of the upper (resp. lower) edge in Q satisfies that it is higher (resp. lower) than the upper (resp. lower) in P, then Q can be obtained from P by the addition of a redundant and deletion of a single edge, and the same conclusion applies.

Now assume that $T(P) \not\subset T(Q)$. Then if $T(P) = (a_1, b_1)$ and $T(Q) = (a_2, b_2)$ then either $a_1 < a_2$ or $b_1 > b_2$ (if it is both then $T(Q) \subset T(P)$, and by the above argument $P \not<_F Q$). If $a_1 < a_2$ the jump sequence of P will dominate the jump sequence of Q and therefore $P \not<_F Q$. If $b_1 > b_2$ the star-jump sequence of P will dominate the star-jump sequence of Q and therefore $P \not<_F Q$. Thus there must be no relation $P \leq_F Q$, consistent with our proposition.

Since every pair P and Q of naturally labeled posets of Greene shape $(k, 1)$ must fall in to one of the four above cases, we have that $P <_F Q$ if and only if $T(P) \subset T(Q)$. □

Theorem 7.1.2 allows us to only consider the tuple representation when comparing naturally labeled posets of Greene shape $(k, 1)$. This in turn not only allows us to quickly determine the relation (if any) between two posets of this type, but achieve a clean representation of the F-positivity relationships between all naturally labeled posets of Greene shape $(k, 1)$, as can be seen in Figure 7.4. Note that this figure is the upper-half of the lattice given by a product of two chains. The lower half of the lattice is not here because the first element of the tuple must be at least two less than the second element for the connections to be meaningful.

7.2 Mixed spine caterpillar posets

Now we will consider more general set of labeled posets. Note that this means we are reverting to our original definition of the jump. Take the class of posets that consist of a single chain with s elements (referenced similarly to posets of the previous section as the *spine*), and k extra elements each with at most a single weak connection to the chain to result in a poset P of size $|P| = s + k = n$. We will call these posets *mixed spine caterpillar posets* for their spines with strict and weak edges *and* visual similarity to caterpillar graphs.

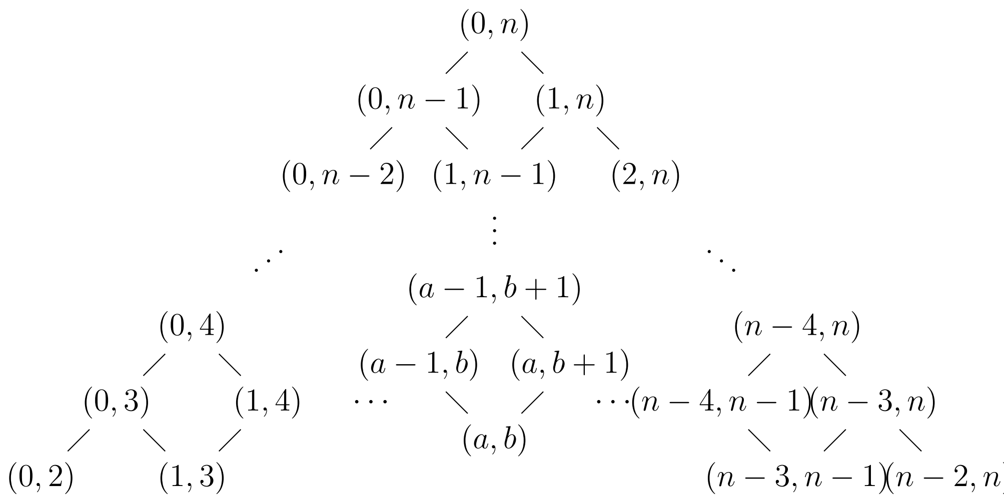


Figure 7.4: The full representation of posets with Greene shape $(k, 1)$, where $|P| = n$

We can encode these posets in a similar way as the naturally labeled posets of Greene shape $(k, 1)$ from the previous section: as a *list* of tuples representing the non-spine nodes. As above these tuples will be of the form (a, b), where either $a = 0$ or $b = s + 1$, or in the case of an unconnected element, both. We will denote this list $\mathcal{T}(P)$, and order the list in lexicographic order. Thus the poset P given in Figure 7.5 would be encoded as $\mathcal{T}(P) = (0, 3), (0, 5), (0, 8), (1, 8), (1, 8), (4, 8), (5, 8), (6, 8)$.

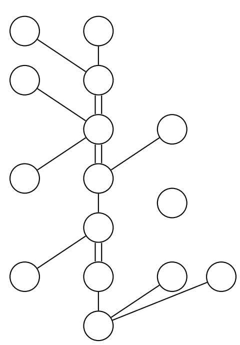


Figure 7.5: An example of a mixed-spine caterpillar poset

The bar involution will be a useful tool in the next two results, including this preliminary lemma.

Lemma 7.2.1. *Say that the mixed-spine caterpillar posets* $(P, \omega) \leq_F (Q, \tau)$ *have* s *elements in their spines and* $|P| - s = k$ *other nodes. We will let* X *denote either* P *or* Q*. Let* α_X *be the number of non-spine nodes that cover an element of the spine in* X*, let* β_X *be the number of non-spine nodes that are covered by an element of the spine in* X*, and let* γ_X *be those elements that are unconnected in* X*. Then the following are true:*

(i) $\alpha_P + \gamma_P \leq \alpha_Q + \gamma_Q$,

(ii) $\beta_P + \gamma_P \leq \beta_Q + \gamma_Q$, *and*

(iii) $\gamma_P \leq \gamma_Q$.

Proof. If we bar-involute the poset if necessary so that the minimal connection on the spine is strict, the only elements with jump 0 will be the ones covered by the spine, the minimal node of the spine, and those elements that are completely disconnected. For example the bar-involution would have to be performed on the poset in Figure 7.5. Thus the first element of the jump sequence of X is $1 + \beta_X + \gamma_X$. By performing the star involution and proceeding similarly, we know that the first element of the star-jump sequence is $1 + \alpha_X + \gamma_X$. Since both sums for P must be less than those for Q, we know that (i) and (ii) hold.

For (iii), we know that $\alpha_P + \beta_P + \gamma_P = s = \alpha_Q + \beta_Q + \gamma_Q$. Thus we know by summing the inequalities in (i) and (ii) that

$$\alpha_P + \gamma_P + \beta_P + \gamma_P \leq \alpha_Q + \gamma_Q + \beta_Q + \gamma_Q.$$

Thus $s + \gamma_P \leq s + \gamma_Q$, and we have (iii). □

We will denote the ith index tuple of $\mathcal{T}(P)$ as $\mathcal{T}(P)_i = (a(p_i), b(p_i))$, hereby referring to the extra node whose connections to the spine are represented by $\mathcal{T}(P)_i$ as p_i. We can obtain from this representation a similar relationship as in Theorem 7.1.2.

Proposition 7.2.2. *Say that P and Q are both caterpillar posets with identical spines, and extra elements that are connected to each spine with either all weak (or equivalently all strict) edges or no connection. Then $P \leq_F Q$ if and only if $\mathcal{T}(P)_i \subseteq \mathcal{T}(Q)_i$ for all $i = 1, ..., k$.*

Proof. Let s denote the number of elements in the spines of P and Q.

($\Leftarrow$) Assume that $\mathcal{T}(P)_i \subseteq \mathcal{T}(Q)_i$ for all $i = 1, ..., k$. That means for each i, the tuple $\mathcal{T}(Q)_i = (a_{Q,i}, b_{Q,i})$ can be obtained from $\mathcal{T}(P)_i = (a_{P,i}, b_{P,i})$ by the deletion of an edge, possibly preceded by the addition of a redundant edge. We know that it is impossible to create bad cycles in this case, because the redundant edge and the original edge will be either both strict or both weak. Thus by Theorem 4.4.1 we have $P \leq_F Q$.

($\Rightarrow$). We will proceed by the contrapositive. Now assume that there exists at least one index $i \in \{1, 2, ..., k\}$ where $(a_{P,i}, b_{P,i}) \in \mathcal{T}(P)$ and $(a_{Q,i}, b_{Q,i}) \in \mathcal{T}(Q)$ are such that $(a_{P,i}, b_{P,i}) \nsubseteq (a_{Q,i}, b_{Q,i})$. Say that the element in P whose tuple is $(a_{P,i}, b_{P,i})$ is p_i and the element in Q whose tuple is $(a_{Q,i}, b_{Q,i})$ is q_i. There are three different ways p_i (resp. q_i) can be connected to its spine of P (resp. Q). Either p_i (resp. q_i) covers an element of its spine, is covered by an element of its spine, or is disconnected. With our ordering, we know that $\mathcal{T}(P)$ and $\mathcal{T}(Q)$ will list tuples representing elements that are covered by an element of respective spines, then those with no connections, then those that cover an element of its spine.

Case 1—p_i is covered by an element of its spine, q_i is not

Say first that p_i is covered by an element of its spine and q_i covers an element of its spine. By the order of the tuples from $\mathcal{T}$, if a tuple in $\mathcal{T}(P)$ of type $(0, b_P)$ where $b_P \leq s$ is compared to a tuple in $\mathcal{T}(Q)$ of type $(a_Q, s+1)$ where $a_Q \geq 1$, then P will have more non-spine nodes that are covered by an element of its spine (β_P) than Q has elements that are covered by or are disconnected from its spine combined ($\beta_Q + \gamma_Q$). By Lemma 7.2.1 (ii) we have $P \not\leq_F Q$. Thus we know that p_i cannot be covered by an element of its spine of P while q_i covers one of its spine.

If q_i is disconnected then $(a_{P,i}, b_{P,i}) \subseteq (a_{Q,i}, b_{Q,i})$ regardless of what $(a_{P,i}, b_{P,i})$ is, contrary to our assumption. Thus there is nothing to check for this case.

Case 2—p_i covers an element of its spine, q_i does not

Say first that p_i covers an element of its spine and q_i is covered by an element of its spine. By the order of the tuples from $\mathcal{T}$, if a tuple in $\mathcal{T}(P)$ of type $(a_{P,i}, s+1)$ where $a_P \geq 1$ is compared to a tuple in $\mathcal{T}(Q)$ of type $(0, b_{Q,i})$ where $b_Q \leq s$, then P will have more non-spine nodes that cover an element of its spine (α_P) than Q has elements that cover or are disconnected from its spine combined ($\alpha_Q + \gamma_Q$). By Lemma 7.2.1 we have $P \not\leq_F Q$. Thus we know that p_i cannot cover an element of its spine of P while q_i is covered by one of its spine.

If q_i is disconnected then $(a_{P,i}, b_{P,i}) \subseteq (a_{Q,i}, b_{Q,i})$ regardless of what $(a_{P,i}, b_{P,i})$ is, contrary to our assumption. Thus our claim still holds.

Case 3—p_i is disconnected from its spine, q_i is not

If q_i is covered by an element of its spine, then by the way that $\mathcal{T}$ orders the tuple we know that P must have more elements that either cover an element of or are disconnected from its spine ($\alpha_P + \gamma_P$) than $\mathcal{Q}$ has elements of the same ($\alpha_Q + \gamma_Q$). By Lemma 7.2.1 we know that $P \not\leq_F Q$. An entirely analogous argument holds if q_i covers an element of its spine.

Case 4—Both p_i and q_i both cover elements of their spines

We assume then that $(a_P, k+1) \not\subseteq (a_Q, k+1)$. This means that $a_P < a_Q$. This however means that there are more elements of $\mathcal{T}(P)$ with first element less than or equal to a_P then there are in $\mathcal{T}(Q)$. There must also be at least one covering edge in the chain between its spine element that is covered by p_i and the corresponding one that is covered by q_i. Say first that this is a strict edge. Thus we have that $\text{jump}(p_i) < \text{jump}(q_i)$. This means that the sum of the jump sequence to the $\text{jump}(p_i)$–th element is greater for P than it is for Q. This means that $\text{jump}(P) \nleq_{dom} \text{jump}(Q)$, and therefore $P \nleq_F Q$. If there is no strict connection, then we are guaranteed a weak one and therefore $\overline{\text{jump}}(p_i) < \overline{\text{jump}}(q_i)$. This means that the sum of the weak jump sequence to $\overline{\text{jump}}(p_i)$–th element is greater for P than it is Q. Thus we have that $P \nleq_F Q$ in the same manner.

Case 5—Both p_i and q_i both are covered by elements of their spines

Now say that both p_i and q_i are covered by an element of their respective spines, and we assume that $(0, b_{Q,i}) \nsubseteq (0, b_{Q,i})$. If we perform the star involution, we will have the case discussed in Case 4, and we know again that $P \nleq_F Q$.

Finally note that there is nothing to check if both p_i and q_i are disconnected from their spines.

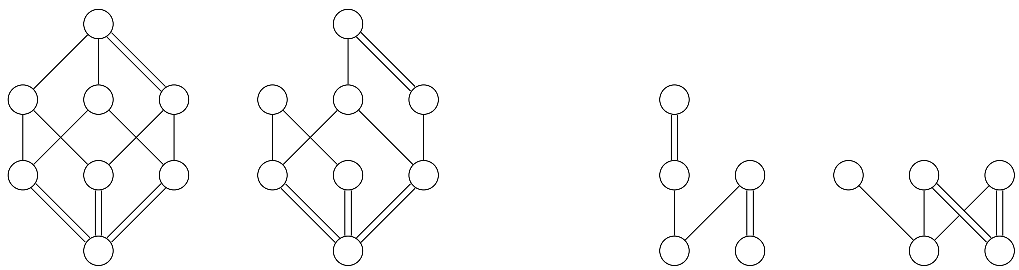
Thus we have in all cases if there exists at least one $j \in \{1, 2, ..., k\}$ where $(a_{P,i}, b_{P,i})$ in $\mathcal{T}(P)$ and $(a_{Q,i}, b_{Q,i}) \in \mathcal{T}(Q)$ are such that $(a_{P,i}, b_{P,i}) \nsubseteq (a_{Q,i}, b_{Q,i})$, then $P \nleq_F Q$. $\square$

Conclusion

In this thesis, we studied the (P,ω)-partition generating of labeled partially ordered sets under the F-basis for quasisymmetric functions, and found methods of determining whether or not the difference between two of these generating functions is or could be F-positive. We showed that if the labels induce all weak or all strict edges in one of the posets we can quickly tell if it is at all possible that the two are related. We showed that there are a number of necessary conditions for F-positivity based off the jump sequence. These conditions are not sufficient to show that two labeled posets are related in the F-positivity order, but since they are necessary they *can* show quickly when two posets are *not* related.

We next considered linear extension containment, which is a sufficient condition for F-positivity. We then considered the operation of adding redundant edges then deleting edges from the Hasse diagrams. By means of less-than sets showed that those posets that are related by linear extension containment are exactly those that can be obtained from one another by redundant-deletion.

We showed how to use known relations to build far larger ones by means of the Ur-operation, a generalized method of combining posets. This is particularly adept at showing relations between posets that are too large to reasonably compute even by means of a computer. This is because the worst case time for the computation grows at least factorially, and combining posets together grows the size of the resultant posets very quickly. For example the relations below $\mathcal{L}(\mathcal{P},\omega) \subseteq \mathcal{L}(\mathcal{Q},\tau)$ and $(P_i,\omega_i) \leq_F (Q_i,\tau_i)$ are quickly done on a computer:



$(\mathcal{P},\omega)$ $(\mathcal{Q},\tau)$ (P_i,ω_i) (Q_i,τ_i)

But $(\mathcal{P}[x_i \to P_i],\Omega)$ and $(\mathcal{Q}[y_i \to Q_i],\Upsilon)$ each have $8 \cdot 5 = 40$ elements and are very difficult even for a computer to show that F-positivity is preserved.

We then showed a possible method for determining whether or not two posets are related in F-positivity order by means of deletion and reversal.

Finally we gave two classes of posets within which our conditions are both necessary and sufficient: those of Greene shape $(k,1)$ and mixed-spine caterpillar posets.

We conclude with potential areas of further investigation. First all of our purely necessary conditions are based on the M-support of the (P,ω)-partition generating functions. Necessary conditions that are based on F-support, M-positivity, or F-positivity would be ideal. Similarly our sufficient conditions are primarily based on linear extension containment rather than F-positivity. Finding sufficient conditions for posets of the latter nature would be closer to our goal. Finally, we intend to further investigate the Ur-operation where the “larger” posets $(\mathcal{P},\omega)$ and $(\mathcal{Q},\tau)$ do not have linear extension containment. Computer testing suggests that the resulting posets under the Ur-operation may be related in F-support containment, but that is in no way confirmed and as just shown computer testing with the Ur-operation quickly becomes intractible.

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