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Launch-Explore-Summarize in High School Calculus

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LAUNCH-EXPLORE-SUMMARIZE IN HIGH SCHOOL CALCULUS

by

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ABSTRACT

Current research on high school calculus instruction indicates that students often possess a procedural knowledge of differentiation and integration as opposed to a conceptual knowledge (Orton, 1983; Ferrini-Mundy & Graham, 1994). Given the prominence of traditional lecture and textbook-based calculus classes in the United States, students are not always given the opportunity to expand their conceptual knowledge of essential calculus concepts. This project introduces calculus students to a more active and communal method of teaching: Launch-Explore-Summarize (LES) (CMP, n.d.). This methodology places students at the center of their learning, and emphasizes inquiry-based thinking during a class. Specifically, two LES lessons are designed and taught in high school calculus classes in order to offer students a conceptual basis for thinking about differentiation and integration. Lesson data and student feedback are discussed in relation to traditional calculus instruction, and ultimately offer insight into the potential effectiveness of LES in high school calculus. The study finds that LES lessons are effective in collaboratively engaging students with calculus material, and that LES is largely effective in helping students conceptually learn differentiation and integration. Lastly, it finds that traditional calculus teachers are skeptical of LES-based curricula, and that these viewpoints contrast with student perceptions of LES.

CHAPTER 1: INTRODUCTION

As students progress through high school, they encounter a variety of teaching styles and methodologies. These teaching philosophies generally fluctuate between high school subjects, such as mathematics and history, but can also differ within a subject. For instance, two algebra teachers may differ in the way they present material and structure their classrooms. One teacher may emphasize the procedural components of algebra while the other focuses on the underlying geometric concepts of algebraic reasoning. Regardless of methodology, it is important to recognize that such distinctions exist and affect student learning and engagement (Stigler & Hiebert, 2004). As a future mathematics educator, I find interest in gaining a firsthand account of how students perceive different forms of mathematics instruction and how well such instruction performs in terms of conceptual understanding.

This research project focuses on the comparison of traditional methods of teaching high school calculus to a non-traditional method known as the Launch-Explore-Summarize method of instruction. This non-traditional method places students at the center of their own learning as students work in a collaborative environment to build and explore conjectures, relate their findings to previously learned material, and discuss the implications of their results in a broader mathematical context. This structure is vastly different from traditional methods of lecture and decontextualized problems.

Launch-Explore-Summarize

The Launch-Explore Summarize (LES) method of mathematics instruction was developed by the Connected Mathematics Project (CMP) from Michigan State University (Connected Mathematics Project, n.d.). This lesson structure is divided into three phases: the Launch phase, the Explore phase, and the Summarize phase. While each phase relates to one another and the broader mathematical content in the lesson, they individually contain specific components that augment the overall structure.

In the Launch phase, students are given the opportunity to access the lesson material in an appropriately contextualized manner. The teacher determines the context of the lesson, but it should allow students to engage with the task or problem without revealing key details or components of the activity. For instance, an example of an effective Launch could be a Do-Now task that revisits prior material that will become important for the current lesson. The CMP includes several considerations for the Launch phase, including knowing what prior knowledge a student might need to complete the LES lesson, how the current lesson connects to such prior knowledge, how the LES lesson can be personalized for each class, and how the material can be made accessible to all students.

Once the Launch phase is complete, students move to the Explore phase. In this phase, students actively work on the problem to build mathematical understanding and make connections. It is important for the teacher to make any required materials available to the class and clearly communicate the class structure at this time. For example, are students collaborating in pairs or groups, or working collectively as a class to solve a problem? Once these roles are established, the majority of learning and exploration is the students' responsibility. The classroom teacher surveys the room, answering questions and offering appropriate scaffolding to students, while also preparing for the final phase of the lesson. Because groups may be at different stages of problem solving, the Explore phase is an excellent time to differentiate instruction for specific groups of students. Such differentiation can include effective selecting and sequencing of student responses (Smith & Stein, 2011), with the teacher purposely choosing specific groups to present their work in a logical order.

As part of the Explore phase, students are responsible for transforming their mathematical discoveries into a presentable format in order to share their findings with the whole class. While visual representations, such as a poster or digital file, are encouraged if appropriate, it is worth noting that such presentations may also be verbally explained or physically modeled, among other options. This basic structure offers classroom teachers flexibility in how they organize the lesson, and may be molded to align with the needs and community of the classroom. Giving each group of students ample time to complete the task is also important, and having additional questions available for students who progress at faster or slower paces than anticipated is recommended to facilitate a solid LES lesson (CMP, n.d.). Once all groups are at an appropriate point in the lesson and are ready to share with the class, the Summarize phase begins.

The Summarize phase is the final phase of an LES lesson and is used to build an overall class understanding of the problem. In this phase, students share their visuals or explanations from the Explore phase and manage questions or comments from other groups of students. Because each group may have been at different parts of the problem in the Explore phase, it is important for each presenting group to explain what thought processes were used in problem solving. Students may also experiment with other groups' problem solving strategies to test for consistency and correctness, given that they may be different from their individual strategies. This structure ultimately helps the entire class understand the essential concepts of the lesson, as it recapitulates students' work from the previous phases.

The teacher is responsible for facilitating an effective discussion in the Summarize phase (CMP, n.d.). As previously mentioned, a purposeful selecting and sequencing of presentations aids in this process, as students can hear different ideas about the problem in a logical order (Smith & Stein, 2011). Throughout this process, the classroom teacher must be prepared to ask and answer questions that may not be inherently obvious to the problem (CMP, n.d.). For instance, if an LES lesson is being used to introduce a new topic to the class, the teacher should be ready to ask questions that connect the lesson to the new topic. Additionally, time limitations remain a factor in the Summarize phase. When orchestrating the full class discussion, the classroom teacher will have to make decisions on which questions and conjectures need to be answered at the present time and which can be left until a future class period (CMP, n.d.). Such a practice is important to solidify student understanding of the problem and concepts associated with the lesson. Through effective planning and implementation of the Launch phase, Explore phase, and Summarize phase, the overall LES lesson is designed to provide an interactive and rich mathematical learning environment for students.

Why Calculus?

The goal of this research project is to test the effectiveness of LES lessons in high school calculus. Much of the current literature on the LES method is either dated (e.g. Hirsch, Coxford, Fey, & Schoen, 1995), focused on middle school students (e.g. Karatas & Baki, 2013), or focused on other subjects outside of calculus such as geometry (i.e. Halat, Jakubowski, & Aydin, 2008). There is little to no research done on its effectiveness in high school calculus. Since calculus is an essential part of many high school mathematics curricula, it is important to analyze the current state of calculus education and how LES could be used in this process.

Current literature suggests that in the calculus class, lecture continues to be the main form of instruction (Larsen, Glover, & Melhuish, 2015). Additionally, there are few research studies that explore actual teaching practices in calculus classrooms (Larsen, Marrongelle, Bressoud, & Graham, 2017). Consequently, there is a significant lack of understanding on differing methods of calculus teaching and how these methods affect student learning. This deficit is concerning and needs to be addressed. In an effort to minimize the literature gap, this project aims to understand the structure of current calculus education and provide evidence-based suggestions for improving calculus instruction.

CHAPTER 2: LITERATURE REVIEW

The literature review begins with a broad overview of calculus instruction, followed by a more specific analysis of differentiation and integration instruction. This analysis will explicitly review instruction on the power and product rule for finding derivatives as well as instruction on integration as total area under a function with respect to the *x*-axis. Following this discussion, the literature on the LES method of instruction in the context of general problem-based learning strategies will be explored. This exploration will also include literature on the specific question types used throughout an LES lesson, and how these question types relate to student responses. Finally, the LES method will be discussed in context with current calculus instruction and how this connection leads to my research questions.

Current Calculus Instruction

Before beginning a broad analysis of calculus instruction, it is important to note that much of the available literature is not focused on high school courses. Instead, calculus research is largely conducted in college-level courses rather than the high school classroom (e.g. Wagner & Sharp, 2017; Bressoud, 2015; Aspinwall, Shaw, & Presmeg, 1997). While a discussion of calculus instruction in the university setting helps understand calculus teaching as a whole, it is concerning that only a few studies centered on high school calculus exist. This project will work to bridge this gap in the literature.

Calculus is the study of limits, derivatives, and integrals. Though these are core topics of calculus, the literature suggests that students struggle with developing a conceptual understanding of them. Epstein (2013) describes the Calculus Concept Inventory (CCI) which is a test of students' conceptual understanding of fundamental topics in differential calculus. When administered in the United States and several other countries around the world, nearly every country reported that students lacked a conceptual understanding of calculus. Although Chinese students performed significantly better on the CCI, Epstein was able to discredit the misconception that Chinese students are adept at drill and practice procedures, noting that calculus instruction in China is more fundamentally based than instruction in the United States.

Teaching and Learning Differentiation

Literature on Students' Understanding of the Derivative

Although the concept of derivatives is a focal point in the study of calculus, student understanding of differentiation is not holistically strong. Rather, students are much more adept at the procedural skills involved with differentiation as opposed to the central underpinnings of the concept. Orton (1983a) initially observed this trend, as he concluded that students were largely capable of using procedures to find derivatives but encountered many more difficulties when they needed to apply their knowledge of the derivative to problems involving rates of change and the limit definition of the derivative. Ferrini-Mundy and Graham (1994) also observed this trend. For instance they found that students could not navigate different representations of the derivative as well as they could calculate derivatives using formulas.

Additionally, the literature suggests that students have trouble interpreting the graphs of derivatives and connecting these graphs to original functions. Nemirovsky and Rubin (1992) worked to understand how students perceived the relationship between a function and its derivative; however, they attempted to do so from the student's point of view using a process similar to Tall and Vinner's (1981) framework of concept images and concept definitions. Given the logic and complexity involved with learning mathematics, Tall and Vinner introduced a moldable framework for analyzing student understanding, both at the present moment and as new information becomes available to the student. For a given concept, they regard a concept definition as "a form of words used to specify that concept" (Tall & Vinner, 1981, p. 152). It is important to note that Tall and Vinner explicitly distinguish between a personal concept definition and a formal concept definition. A personal concept definition is that of a particular student at a particular time, and it may change as the student learns new information about the concept. Conversely, a formal concept definition is the widely accepted understanding of a concept by the mathematical community (Tall and Vinner, 1981, p. 152).

In Nemirovsky and Rubin's (1992) study of derivative graphs, the researchers wanted to compare the perceptions and understanding of derivative graphs from students' personal concept definitions to that of a formal concept definition. In other words, the goal of the study was to determine how well students' understandings of derivative graphs actually meshed with correct interpretations of the concept. Attributes involving the graphs of derivatives, including slope, rates of change, and relations to original functions, were placed in various contexts. For many of the students involved with the study, it was difficult to connect given information about a function with its derivative, and to effectively utilize these tools to solve application problems.

Despite the trend of students having a greater procedural knowledge base as opposed to a conceptual knowledge base, it is worth mentioning that the literature is lacking comprehensive studies of students' understanding of the derivative. As Larsen et. al (2017) notes,

> The research on student understanding of the derivative is characterized by small, detailed studies of students' thinking as they solve problems designed to probe their ability to carry out derivative computations, think about graphical representations, and make connections between multiple representations of the derivative (p. 535)

Given the literature gap associated with the teaching and learning of differentiation, it is difficult to gain a comprehensive understanding of its current state in calculus education. In an effort to build some comprehension, it is worthwhile to review example lessons of core differentiation topics. It is also noteworthy that some of the following activities occur outside of high school classrooms. However, their setup and levels of student understanding are comparable to that of a high school classroom.

Examples of Differentiation Tasks

Wagner and Sharp's (2017) activity focused on the relation between secant and tangent lines. This activity was presented to a group of 61 first-semester calculus students at a public university, and employed inductive reasoning to build connections between secant and tangent lines. The goal of the activity was for students to use GeoGebra software to graph secant and tangent lines, observe and change the slope of these lines, and build conjectures about how they compared. Ultimately, students began to offer suggestions about how to approximate the slope of a tangent line with a secant line, including bringing the intersecting points of the secant line closer together to mirror that of the tangent line. Wagner and Sharp concluded that, even at the end of the semester, a significant percentage of students were able to articulate the relationship between secant and tangent lines in a conceptual and precise manner.

Although the Wagner and Sharp activity is just one example, it is important to understand its purpose in teaching differentiation. Such an approach was student-centered, interactive, and discovery-based. While the researchers did not compare their findings with that of traditional instruction, they were able to assert that 74 percent of students could correctly and conceptually describe secant and tangent lines, and that 63 percent could do so on the final exam twelve weeks after the lesson. This activity shows the potential for students developing a conceptual understanding of an essential derivative topic when taught in an active manner.

While the Wagner and Sharp activity is encouraging, there are other differentiation lessons that fall under a more traditional lens. Consider Hurwitz's (2001) activity involving the product rule for derivatives. Students in this activity had to determine if the derivative of a product is the product of the derivatives, as is the case for similar structures involving limits or square roots. The lesson proceeded with a mix of conjectures and examples of derivatives with and without the product rule formula, as students worked to decipher if their initial conjectures were true. Although there was some conceptual work present in this activity, the majority of the lesson focused on formula manipulation and testing rather than understanding the concepts associated with the product rule. Such an activity offers support as to why students are more comfortable with procedural knowledge instead of conceptual knowledge, as the literature has concluded.

Textbook Explanations of the Derivative

Textbooks play a critical role in calculus classes and how the derivative is often presented. Nicol and Crespo (2006) note that, especially in North American classrooms, textbooks are a key component of mathematics education. Specifically, textbooks often dictate what material is taught in the classroom, how it could be taught to students, and when it could be taught in the curriculum (p. 331). Given the powerful role that textbooks can play in calculus courses, it is worthwhile to discuss how they introduce and explain derivative concepts.

My study focuses on how three textbooks present the power rule and product rule for calculating derivatives. The three chosen textbooks are 1) *Calculus: Ideas and Applications* (Himonas & Howard, 2003), 2) *Essential Calculus* (Stewart, 2007) and 3) *Calculus: Early Transcendentals* (Rogawski & Adams, 2015), which could all feasibly be used in a high school calculus course. All three include chapters on the essential components of calculus, such as limits and continuity, differentiation, and integration, introduced in that order. After examination, all three textbooks are highly similar, especially in their explanations of the power rule and product rule for derivatives. They each first introduce the derivative in the context of average and instantaneous rates of change before formally defining the derivative as a limit. The order of presentation shifts from here, but all three books cover basic derivative formulas, the chain rule, implicit differentiation, and related rates.

There is a common theme in each of the books' respective sections on the power rule: each book offers the reader some examples that use the power rule in order to see a pattern before presenting the generalized rule in a highlighted box. The product rule is nearly identical, as each book presents the generalized rule in an emphasized box. Textbook 2 includes an example of why the derivative of a product is not the product of the derivatives. After these explanations, students are supplied a series of problems where they use the power and product rules to find basic derivatives.

A brief analysis of calculus textbooks support the literature discussed earlier: students can calculate derivatives using a variety of procedural rules, but struggle when needing to conceptualize or apply knowledge of the derivative. Given the procedural emphasis placed on derivatives in each of the textbooks, and the role of textbooks as explained by Nicol and Crespo (2006), it is understandable why such trends exist in the literature. Overall, the teaching and learning of derivatives lacks an applied and conceptual basis, which is only furthered by activities and explanations rooted in procedure.

Teaching and Learning Integration

Literature on Students' Understanding of the Integral

Much like the literature on students' understanding of differentiation, the current literature lacks a substantial amount of research on high school students' understanding of integration. Orton (1983b) was again one of the first to study this topic. He presented a series of integration tasks to both high school and college students, which included questions on limits of sequences, area calculations, geometric areas under graphs, integration procedures, and applications of integration. Like his differentiation results, Orton found that high school and college students tended to perform similarly on the integration assessment.

Further, the students were more successful in completing more procedural tasks, such as carrying out integration or calculating areas, as opposed to more conceptual tasks, such as the relationship between sequence limits and area or the fact that the integral of sums equals the sum of integrals. Orton also concluded that, in the context of a problem, many students knew what to do to solve the problem but were unsure why they were using such a procedure. Orton's findings align with traditional calculus instruction, as the emphasis on procedural understanding often outweighs a solid conceptual basis.

Although not an explicit study like Orton's, Tall (1992) also discussed the difficulties that students face in learning calculus. Initially, Tall separated calculus into two realms: informal calculus and formal analysis. In the informal calculus session, Tall included general ideas of rates of change, rules for differentiation and integration, and areas and volumes as applications of integration. Conversely, the formal analysis section portrays calculus with an emphasis on completeness. Such topics include formal definitions of limits, continuity, differentiation, Riemann integration, and the fundamental theorem of calculus (p. 13). Tall also notes that, while instruction obviously differs from classroom to classroom, there is a general dissatisfaction with the structure of calculus courses. Specifically, he echoed the common theme of the literature, arguing that a conceptual understanding of core calculus topics was not the focal point of instruction.

With regard to integration, Tall meticulously explored the concept of limits and how they are used to define much of calculus. Specifically, he noted that limits tended to be conceptually difficult for students. Language such as “tends to” and “as small as we please” often interfered with formal concepts, and limit processes not done by arithmetic or algebra created a mysterious realm for students (p. 14). Tall also summarized how students often struggle with connections between concepts. Given the novelty of the essential topics in calculus, Tall explained that students either “reconcile the old and the new by re-constructing a new coherent knowledge structure” or “keep the conflicting elements in separate compartments and never let them be brought simultaneously to the conscious mind” (p. 15). Although the former is a solid learning strategy, its difficulty often pushes students to adopt the latter. Overall, with regard to integration and calculus instruction in general, Tall’s summary again confirms the theme of the literature that students struggle more with conceptual notions of key calculus topics.

Before exploring examples of integration instruction, it is helpful to discuss how students perceive the conceptual material of integration. Rasslan and Tall (2002) conducted a study with 41 high school students regarding the definition of the definite integral in an effort to answer this exact question. The students were given a short questionnaire with aspects of the definite integral concept, including calculations, connections to area and total quantity, and definitions. Specifically, Rasslan and Tall asked students to explicitly define the definite integral over a closed interval in the final question of the assessment. Out of the 41 students, 26 gave no answer to the question. The remaining 15 students either defined it as a procedure involving antiderivatives, formulas for definite integrals, or the area between the graph and *x*-axis. Such a disparity led Rasslan and Tall to conclude that the majority of students struggle to apply meaning to the definite integral and have difficulty interpreting and applying this concept in context (p. 96).

More recently, however, Sealy (2014) has developed a Riemann Integral Framework which decomposes total quantity problems into Riemann sums involving four layers: product, summation, limit, and function. Her methods were an attempt to help students observe and utilize the underlying structure of integration as opposed to limiting focus to the integral as area under a curve. Sealy presented a variety of different total quantity problems to students to engage them with the Riemann Sum Framework and noticed similarities in structure between the tasks. She found that students largely understood each of the problems and could utilize the framework effectively. Sealy noted that this was a surprising result, but also commented that further research was necessary to fully assess the Riemann Sum Framework. Such an approach could be beneficial in helping students ascertain the essential components that define the integral, as well as offering tools to solve a multitude of contextualized integration problems.

Examples of Integration Tasks

As in the differentiation section, a review of integration tasks is helpful in understanding how teachers introduce the integral to students. Jones' (2013) introductory task allows students to build a meaningful understanding of area without the notational distractions involved with the integral. Jones placed his students in the context of a bursting water pipe leaking at a constant rate. This constant rate was a deliberate choice, as it allowed students to focus on area and not a complicated function. Since the water leaked in liters per minute, students also needed to make the conceptual leap of area as two-dimensional to volume which is three-dimensional. Students then worked on an extension of the pipe problem where the water leaked at varying rates. Again, no function or graph was given, but the students used a table of data to generate a sequence and approximation for the total amount of spilled water over time intervals. Finally, students connected this result to a graph of the situation and discussed how to improve methods of approximation. The activity concluded with a discussion about how limiting Riemann sums relate to the integral, all in the general context of the problem.

The Jones task is an excellent example of effectively introducing integration, as it allowed students to explore and become familiar with the concept of the integral without being limited by notation and formality. While the previously discussed literature encourages an understanding of this material, building a solid conceptual foundation will ultimately lead students to this point. Jones' task is similar to Sealey's (2014) Riemann Integral Framework, as it decomposes the Riemann sum into finer pieces in the context of a tangible situation. Students can also use their intuition and experimentation to build the integral definition rather than navigating a formal presentation from the beginning.

For comparison, consider Ilaria's (2014) activity that also introduced students to integration. Ilaria spent a week systematically introducing his students to the integral as the area under a curve and above the *x*-axis. Students initially made conjectures about how to calculate this area before drawing rectangles under the curve. For homework, students recreated this process using a larger number of rectangles. In the next class, students utilized different techniques for approximating areas, including the left and right hand methods. The week continued with a discussion regarding a generalized process for any number of rectangles before making the conceptual jump to limits of infinite rectangles. The final class periods were used to subtly introduce antiderivatives to students, as they were tasked with finding a function whose derivative yielded the class function. The class finished with a summary of their findings from the week before exploring the topic further in future lessons.

Although Ilaria's lesson was more concrete in terms of functions that Jones', it did allow students to build their own understanding of the integral as well. A balance between formal notation and conceptual understanding allowed students to employ intuition before formulas in many of the class periods. It was only at the end of the week that the class transitioned to conjectures about antiderivatives and limits of Riemann sums, although it appears this process was not as conceptual as Jones' lesson. Despite their differences, both Jones and Ilaria's lessons share some common themes. They both utilized student intuition and exploration as the primary learning method, while also emphasizing student conjectures throughout the class period. Allowing students to build their own connections and schemas in early integration lessons coincides the literature as well. In the next section, this approach will be examined in contrast with that of the three calculus textbooks from earlier.

Textbook Explanations of the Integral

As with the section on differentiation, reviewing how integration is introduced to students in textbooks is also a worthwhile endeavor. Recall that Nicol and Crespo (2006) found that textbooks are largely responsible for what and how content is taught in the classroom. The introductory section on integration from the same three calculus textbooks will be discussed: 1) *Calculus: Ideas and Applications* (Himonas & Howard, 2003), 2) *Essential Calculus* (Stewart, 2007) and 3) *Calculus: Early Transcendentals* (Rogawski & Adams, 2015).

All three textbooks place their chapter on integration after the chapters on differentiation. They each present properties and applications of derivatives, such as optimization and graph sketching, before beginning the fondly labeled chapters "Integration," "Integrals," and "The Integral," respectively. All three textbooks contain a one paragraph introduction that prefaces the chapter. Interestingly, every author includes an informal definition of the integral as an area, and also makes reference to the fundamental theorem of calculus which is described as a "connection" between differential and integral calculus.

From here, the textbooks differ in the order of content. Textbooks 2 and 3 begin with a section on approximating areas under a curve using rectangles. They also include information on total distances, similar to the context of Jones' (2013) activity from earlier. Although both books make the transition from finite approximations to infinitely many rectangle approximations, Textbook 3 includes a brief section on summation notation. The books conclude with a series of practice problems involving area and distance calculations before beginning the next section on the definite integral.

Textbook 1 differs in its sequence of integration topics. Its first three sections of the chapter are on indefinite integrals and rules for finding antiderivatives, integration by substitution, and finally integration by parts and partial fractions. It is not until the fourth section that area and the definite integral are discussed, again by rectangle approximations and Riemann sums. Textbook 1 briefly describes the fundamental theorem of calculus at this point, although a fuller explanation is given in a subsequent section.

Similar to the differentiation sections, the textbook sections on integration are also alike. They all express the integral as an area under a curve obtained by a limiting Riemann sum of rectangular areas, and basic properties of integration, including integrals of constants and the linearity of integrals, are also found in each textbook.

Such a structure contributes to the existing literature on integration. The integral tends to be portrayed as an area, and procedures of integration compose a significant portion of each chapter. Finally, aside from the brief section in the Textbook 3, these textbooks fail to fully explain Riemann sums. Although they each heavily rely on Riemann sums to define the integral, it appears that each book assumes that students are familiar with this topic. Sealy’s (2014) framework for Riemann sums suggests otherwise, indicating another disparity in procedural and conceptual understanding. Given the hefty influence that textbooks have on mathematics classrooms (Nicol & Crespo, 2006), it is likely that such a textbook sequence only furthers conceptual difficulties that students have when learning integration.

Launch-Explore-Summarize (LES)

In the preceding sections, the literature focused on the current state of calculus instruction as a whole, methods of instruction for integration and differentiation, student perceptions of integration and differentiation, and the role of textbooks in this cycle. In what follows, a further investigation of the LES method of teaching is conducted, including research regarding this methodology, questioning types included in an LES lesson, and LES implications for calculus instruction.

Research from the Connected Mathematics Project

Although the structure of LES instruction was explored in chapter one, it is important to place this teaching method in context of other active learning strategies. The Connected Mathematics Project (2018) published a literature review of research on their resources and lesson design. The data that is available in the review contains information on the entire project and not strictly LES; however, these data provide encouraging results. The project reported that students in CMP classrooms achieve greater conceptual gains in areas of mathematical modeling, reasoning, and articulation, and that these progressions continue from middle school to high school. Additionally, students' perceptions of mathematics were more positive when exposed to continued CMP instruction. Finally, CMP students performed as well or better than non-CMP students on measures of procedural skills, indicating that the method is still effective in maintaining these requirements.

The CMP research report also contains information on how teachers perceive this method of instruction. CMP classes tended to place a greater emphasis on student communication of mathematical ideas as opposed to traditional classroom structures. Students also reported higher levels of satisfaction in these mathematics classes compared to students in non-CMP classes. Once teachers had the opportunity to observe the types of mathematics students were capable of doing in CMP classes, they preferred to continue with CMP instruction instead of reverting back to traditional teaching.

Despite the differences in the planning and implementation of CMP instruction methods and traditional mathematics teaching, teachers continued to improve with appropriate levels of professional development. Overall, general instruction techniques promoted by the CMP, including LES, have positive trends for students and teachers. Classes are more focused on student communication and conceptual understandings, and promote positive opinions of mathematics and mathematical learning.

Connections to Problem-Based Learning

In an overview of problem-based learning (PBL), Savery (2006) defined PBL to be "an instructional (and curricular) learner-centered approach that empowers learners to conduct research, integrate theory and practice, and apply knowledge and skills to develop a viable solution to a defined problem" (p. 12). Within PBL environments, a significant amount of the learning responsibilities are placed on the students as they engage with the problem. In a collaborative effort, students utilize their current knowledge and experiences to make progress on the problem, often employing a wide range of disciplines and skills.

The problems themselves must also be designed in such a way that structure is flexible. Because PBL places students at the center of learning, it is important that multiple solution paths to the problem exist. Savery (2006) noted that, when a problem is well-structured, students are actually less interested and motivated to develop solutions because a clear path is already in place (p. 13). Essentially, problems with a defined sequence of steps or structure are less interesting for students because they can simply follow a procedure to obtain an answer. Finally, upon completion of a PBL activity, the class should discuss what concepts were needed to solve the problem and what concepts had to be learned to make progress on the activity. In the midst of a problem, students may not actively be recording which strategies or concepts they utilized or needed to learn, so an analysis of problem solving can be incredibly beneficial for the class.

Given the structure of its activities, there are several similarities between LES instruction and PBL. Both introduce students to an activity during a Launch phase, which must be carefully designed to spark interest and curiosity while also not revealing too much about the lesson. Students then work collaboratively to answer the problem in an Explore phase. Finally, students discuss what conjectures and solutions they obtained during the Explore phase in the full class Summarize phase.

Due to the lack of research on LES instruction specifically, it is helpful to briefly review the literature on the effectiveness of PBL instruction because the instructional methods are similar. Barron and Darling-Hammond (2008) recently reviewed the literature surrounding the overall results of PBL as part of a larger literature review on inquiry-based learning. The first significant finding of PBL instruction is that students performed at an equal or higher level on procedural skills. However, PBL students also showed significant improvement on measures of critical thinking and knowledge transfer when compared to students of traditional instruction. PBL students also demonstrated an increased ability to define and solve problems as well as articulate and support claims and arguments. Finally, PBL instruction was often more effective in teaching students who otherwise struggle in traditional environments, as it not only provides a new context to learn in, but a positive description and method of learning as well. Overall, the literature supports the overarching success of problem-based learning. Due to the similarities between PBL and LES, it is plausible that the PBL trends will continue for LES instruction.

Connections between LES and the Common Core

While there are no universal standards for non-AP calculus, the Common Core (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) outlines eight standards for students' mathematical practice (SMPs) that apply to all levels of K-12 mathematics instruction. These standards are listed in Appendix A. Specifically, LES instruction allows students to employ all of the SMPs, depending on the design of the lesson. The LES structure is specifically designed to encourage collaboration and the construction of viable arguments in the context of precise pattern recognition. Students also utilize appropriate tools and models to increase their understanding of the activity and related concepts. Despite the lack of calculus Common Core standards, it is clear that LES instruction meets the standards for mathematical practice.

Questioning and Student Responses in LES Environments

A significant component of LES instruction is effective questioning and student responses. In an effort to maximize the benefits associated with LES teaching, consider Smith and Stein's (2011) framework, *5 Practices for Orchestrating Productive Mathematics Discussions*. This framework includes a variety of question types that correspond to different levels of cognitive demand. In the context of this project, cognitive demand is defined to be a measure of how much effort students need to make to understand a concept. Smith and Stein concur, noting the following:

> Teachers can induce students to think harder about cognitively challenging tasks. Good questions certainly help. They can guide students' attention to previously unnoticed features of a problem or they can loosen up their thinking so that they gain a new perspective on what is being asked. (p. 62)

The framework also provided a foundation for the types of responses that students ideally supply. Table 1 offers a brief classification of how the question types corresponded to student comment types along with their associated level of cognitive demand. Rationale for these choices is explained in the following paragraphs.

Table 1

Relationships between Cognitive Demand, Question Types, and Student Responses

Level of Cognitive Demand	*5 Practices* Question Type	Student Comment Types
Lowest	Gathering Information	General Response
Intermediate	Probing Thinking	Active Thinking
Highest	Exploring Mathematical Meanings/Relationships Extending Thinking	Mathematical Inquiry

Because LES lessons involve different levels of thinking and cognitive tasks, it is necessary to divide question types and student responses among these levels. The levels also correspond to Smith and Stein's (2011) outline of good questions in the *5 Practices* book. The first and least demanding questioning type is Gathering Information, which is used to elicit immediate answers, facts, or procedures. Answers to these questions do not require a significant level of thought, and are used primarily for a teacher to learn about students' current state. Consequently, students tend to give General Responses to Gathering Information questions.

Probing Questions are the next level of question types, as they require more justification and thought when students respond. Probing Questions ask students to articulate, elaborate, or clarify their ideas, often in the context of a mathematical problem. In turn, students respond in a more active manner since they have to justify their thinking for a given question or conjecture. As a result, Probing Questions are often effective introductions to higher question levels, and regularly result in Active Thinking student responses.

The next two levels of questioning are Exploring Mathematical Meanings and/or Relationships and Extending Thinking. These question types are highly related because they both invoke some of the highest thinking that students experience during an LES lesson. Specifically, Exploring Mathematical Meanings/Relationships questions are used to connect mathematical ideas and suggest underlying mathematical structure to students. Extending Thinking questions take this mathematical structure and connect it to other mathematical foci or similar situations. Both question types ask students to think deeply about mathematics, and often prompt students to exhibit Mathematical Inquiry during the lesson.

The final question type from Smith and Stein’s framework is Generating Discussion questions. Although these types of questions can involve each of the other question types already discussed, the major difference is that Generating Discussion questions invoke responses from multiple members of the class. These student responses are usually to one another, and in turn create a rich conversation in the class.

These question types are revisited in chapter four, as there were essential for data analysis. A stronger connection between question types and student responses is also built in chapter four. However, it is helpful to presently introduce Smith and Stein’s framework as a foundation for the research questions and analysis of this project.

Conclusion

Through a careful analysis of calculus instruction and student learning, several significant literature trends emerged. Overall, calculus instruction tends to be traditionally taught using lecture and procedural approaches, and there is not a large amount of research dedicated to high school calculus instruction as a whole. Differentiation and integration are two major topics in calculus, yet students tend to struggle with conceptual understandings of this content. Students are adept, however, at the procedures involved with derivatives and integrals. Textbooks also perpetuate the literature trends, as they are heavily used in American classrooms. An analysis of textbooks revealed that instruction is mostly procedural and decontextualized, indicating a disparity between them and what has been shown to produce meaningful student learning.

The literature also contains research on problem-based learning and the Launch-Explore-Summarize method of instruction. While LES lacks significant research, PBL has been shown to be effective in meeting conceptual goals of mathematics classrooms. It also engages students on multiple levels, including problem-solving, collaboration, and critical thinking. Since LES is similar to PBL, it is expected that the results of LES instruction will be similar as well.

Finally, the literature on effective questioning types is helpful in augmenting the structure of LES lessons. Specifically, connections are made between these question types and student cognitive demand. In turn, it is possible to relate student cognitive demand, teacher question types, and student responses.

CHAPTER 3: RESEARCH QUESTIONS

The LES method of instruction is structured to maximize student engagement and problem solving during a lesson. Students are at the forefront of the learning, as they actively work with manipulatives, technology, and peers to understand and explore a given problem. LES lessons are also naturally adjustable, as each phase of the lesson can be adapted to meet a variety of students. Given the current lecture emphasis in high school calculus, both students and mathematical educators will benefit from research on the effectiveness of LES lessons in high school calculus.

Motivation for the Project

Each summer, Brown University sponsors Brown Summer High School. This program allows local Providence high school students to attend a summer session at the university. Students choose two of four classes in either history, English, science, or mathematics, and attend their two hour classes each day for three weeks. The history, English, and science classes are taught by master level Brown students as part of a yearlong certification program. However, no master's program exists for mathematics at Brown, so this portion of Brown Summer High School is instead filled by undergraduate students.

In a cooperative effort between Brown University and Vassar College, prospective mathematics educators apply to the Teaching Experience for Undergraduates (TEU) program. This program selects undergraduate students from several small liberal arts universities and offers them the opportunity to learn and teach as part of Brown Summer High School. Each TEU participant attends a pedagogy course throughout the six week program and teaches in a team of three in one of the Brown Summer High School math classes.

As part of the pedagogy course, TEU participants learn and practice using the LES method of instruction. Given that there are no curriculum requirements for Brown Summer High School classes, TEU students are responsible for choosing and implementing their own lesson topics. Although the LES method is the core of each Brown Summer High School math class, the actual mathematical topics are decided within individual teaching teams. Coupled with the heavily freshman and sophomore student body of Brown Summer High School, many teaching teams focused their LES lessons on algebraic reasoning.

Research Questions

My participation in the TEU program provided significant motivation for this research project. Using the LES method of instruction in practice was encouraging, but like much of the current literature, implementing it in calculus was not discussed. Consequently, a natural overarching research question was formed. How would LES instruction fair in the calculus classroom, both in terms of student engagement and student learning? To answer this broad question, this project assesses three specific research questions:

1. What types of engagement do students display throughout two high school calculus lessons?
2. How do students perceive the LES method of instruction, both in terms of mathematical learning and engagement?

3. How do teachers perceive the LES method of instruction, both in terms of mathematical learning and engagement?

These research questions help address the aforementioned literature gap, as they primarily test how well the LES structure operates in teaching differentiation and integration in high school calculus.

CHAPTER 4: METHODOLOGY

The Teaching Experiment Methodology

The teaching experiment methodology was chosen due to its structural commonalities with the research project. A significant purpose of this methodology is to "experience, firsthand, students' mathematical learning and reasoning" (Steffe & Thompson, 2000, p. 267). Because the goals of this project are centered on student engagement, discovery, and perceptions of the LES method, it is essential to gather data directly from students. The teaching experiment methodology systematically outlines this process and offers a rich data set for later analysis (Steffe & Thompson, 2000).

The general structure of a teaching experiment is as follows: individuals within a classroom are identified as teaching agents, students, and witnesses, and collectively interact throughout the duration of a lesson (Steffe & Thompson, 2000). For this specific project, I am the teaching agent and the students participating in the project are identified as the teaching experiment students. Finally, the witnesses are the classroom teachers participating in the project.

Prior to beginning a teaching experiment, Steffe and Thompson (2000) indicate the need for research hypotheses. These hypotheses help choose the participants and structure of the teaching experiment, but do not affect the actual teaching of the teaching experiment. Rather, the hypotheses should be forgotten during teaching in order to fully immerse students in the lessons (Steffe & Thompson, 2000). Hypotheses may also be generated throughout the course of teaching, given the possibility for unanticipated situations during a lesson.

Data is collected throughout the teaching experiment. Specifically, each lesson should be documented in some manner to record the events that transpire throughout the duration of the class (Steffe & Thompson, 2000). This may be done using video and audio equipment, with the goal of obtaining a collective set of student interactions, thought processes, and engagements. The recordings should also include interactions between the teaching agent and students.

At the conclusion of a teaching experiment, several processes are completed to analyze the data of the experiment. Steffe and Thompson (2000) note that

> Careful analysis of the videotapes [of the lessons] offers the researchers the opportunity to activate the records of their past experiences with the students and to bring them into conscious awareness. (p. 292)

They continue by emphasizing that this process has the advantage of analyzing lesson interactions that may not have been apparent during the actual teaching. Finally, this analysis must be done in part from the prospective view of a student, as this is necessary in interpreting the significance of student work in completing the lesson (Steffe & Thompson, 2000).

Hypotheses

In keeping with the teaching experiment methodology, this research project has three main hypotheses. The first two hypotheses were developed prior to beginning the teaching experiment while the third was developed throughout the course of teaching. All three hypotheses correspond to the three central research questions of the project, and are also built from existing literature on student engagement and student learning.

Hypothesis 1: *Students will report a higher level of engagement following the LES lessons as compared to traditional instruction, and this pattern will be supported by lesson analysis.*

Although LES instruction is a specific type of teaching, it has similar qualities to problem-based learning (PBL). Current literature suggests that PBL encourages students to take a more active role in the learning process, and that this role supports a more genuine learning experience (Barron & Darling-Hammond, 2008). Additionally, the literature suggests that, because students have a more active voice in their learning, they are able to build mathematical understanding and connections themselves (Schettino, 2012). Given these findings and the similarities between PBL and LES, such a trend will likely continue for LES lessons in calculus. However, caution must be taken to avoid misrepresenting novelty for true mathematical engagement. Because the students in this study are experiencing an atypical class period with an entirely new teacher, it is possible that they engage with the lesson simply for its uniqueness. For this reason, student engagement was analyzed through multiple data sources. This process is discussed in the Data Analysis section at a later point in the chapter.

Hypothesis 2: *Students will positively perceive LES instruction, although student feedback on their engagement will be more positive than student feedback on their mathematical learning.*

Like the first hypothesis, this hypothesis also stems from literature on PBL. First, it is noteworthy that students learning in a problem-based environment learn factual information at a same or better rate than students in a traditional learning environment (Thomas, 2000). However, the literature suggests that PBL students are also more likely to develop and utilize abilities in problem definition (e.g. Gallagher, Stepien, & Rosenthal, 1992), the transferring of problem solving skills to other situations (e.g. Moore, Sherwood, Bateman, Bransford, & Goldman, 1986), and supportive reasoning (e.g. Stepien, Gallagher, & Workman, 1993). Given the similarities of LES and PBL, it is hypothesized that such student benefits would continue during the implementation of LES lessons in calculus.

The second component of this hypothesis addresses direct student feedback, and specific distinctions between engagement feedback and learning feedback. Although it is hypothesized that students will positively react to LES calculus lessons in their entirety, it is expected that there will be more constructive or negative feedback with regard to actual mathematical learning and understanding. Students may be likely to report more positive feedback on engagement with the lesson for many of the same reasons discussed in the first hypothesis. The novelty and change of pace that comes with a new teacher and lesson style may naturally engage students more than traditional class periods, and students may positively report this change.

However, there are limitations on assessing whether or not students actually gain a deeper understanding of mathematical content, at least through direct student feedback. It may be difficult for students to conceptualize whether or not they actually made strides in mathematical understanding immediately following a lesson. Additionally, it may not always be obvious for students to know the purpose of a one or two day interim lesson outside of the research project, even if efforts are made to clarify the lesson's mathematical purpose during teaching. Essentially, although student feedback is a valuable and necessary part of this project, it is important to recognize that students may not be fully equipped to answer such questions immediately following a lesson. For these reasons, the second hypothesis includes the small caveat that student feedback on mathematical learning will be less positive than student feedback on engagement.

Hypothesis 3: *Classroom teachers will find both LES lessons to be effective in engaging students and teaching calculus, but will be speculative about using the method in the future.*

Much like the students, classroom teachers will likely enjoy the novelty of LES instruction. They will also observe that students gain a greater conceptual understanding of differentiation and integration. However, given that classroom teachers generally teach calculus using lecture, it will be difficult for them to imagine regularly using LES instruction. While the rationale behind such a choice may vary between the time constraints of LES, preparation of LES lessons, and perceived student attitudes toward LES instruction, classroom teachers will report concerns with frequent LES instruction.

Participants

In order to select a sample for this research project, contacts were made with seven school districts. Each school was asked whether or not students were offered honors calculus courses and whether classroom teachers would be willing to host a guest instructor for two LES lessons. AP Calculus courses were deliberately excluded in an effort to prevent instructional conflicts with the AP exam. The chosen courses cover the traditional topics of calculus, including limits, differentiation, and integration. Once responses were received from all seven school districts, two were conveniently chosen: Davenport Area High School and Shepherd High School (pseudonyms).

Prior to teaching the LES lessons, students were given an assent form for themselves and a consent form for their parents. The forms are supplied in Appendix B and C, respectively, and offered students an opportunity to not participate in the study if desired. The actual data collection is explained further below, but it is important to note here as it affects the participants in each class.

The Davenport class consisted of 13 students. Of the 13, 11 were female and 2 were male with a mix of juniors and seniors. All but one student in the Davenport class agreed to participate in the study, bringing the final demographic count to 12 students with 1 being male. The Shepherd class was slightly larger as it consisted of 15 students, again with a mix of juniors and seniors. Of the 15, 8 were female and 7 were male. All of the students agreed to participate in the study, and thus between the two schools, 27 students participated in the study. Each classroom had one teacher who was present for both LES lessons. Troy, the Davenport teacher, and Ray, the Shepherd teacher (pseudonyms), had several years of teaching experience in their respective districts.

Lesson Plans

Two LES lessons were designed to capture two of the most prominent topics in calculus: integration and differentiation. Each lesson plan is described in detail in the following sections.

Integration Lesson

The integration lesson was designed and taught in both classes first, as this fell naturally with the Davenport and Shepherd class timelines. This specific lesson was taught in Shepherd during one class period and then in Davenport over two consecutive days. Due to curriculum restrictions, only one class was allotted in Shepherd as opposed to two class periods in Davenport. In conjunction with the current literature, the lesson emphasized a conceptual understanding of integration as total area as opposed to the procedural understanding many students encounter (Rasslan & Tall, 2002). The actual lesson plan is given in Appendix D along with the data table given to students during instruction.

In designing the lesson, the overarching focus was granting students the opportunity to use their intuition to make conjectures about the integral. Prior to instruction, the Davenport students had never seen any integration material. The Shepherd students had begun learning how to solve definite and indefinite integrals but had not yet connected these procedures to a geometric understanding of integration. As a result, the lesson heavily emphasized geometric intuition and was presented knowing each class was unfamiliar with integration as area.

In the Launch phase, students were reintroduced to geometric areas by finding the area of several regular polygons. The polygons were made out of cardboard and included a rectangle, triangle, circle, parallelogram, and trapezoid. A set of all five polygons was supplied to each group along with some meter sticks for measurement and the data table at the end of Appendix D. Figure 1 displays the set of shapes given to each group.

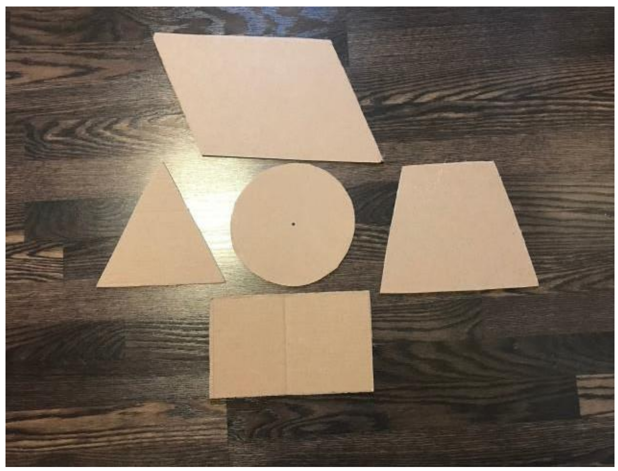

Figure 1. The set of shapes provided to each group.

Not only would these shapes be used for the remainder of the lesson, but they were designed so that students could reconnect with the simplicity of finding the area of regular objects. While students worked, several functions made of string and tape were displayed around the classroom. Each group worked with a specific tape and string function during the Explore phase of the lesson. Once all measurements were made, a class discussion ensued about the simplicity of finding the area of regular shapes and conjectures were made about how the same process could be done for functions. The class discussed what exactly was meant by the “area of a function” and how such a quantity could be systematically calculated across the variety of functions in the room. The class agreed that using the areas of their well-known shapes could be effective in formulating an approximation for the area of each function.

Following the Launch, students took their collection of polygons to one of the functions around the room, tasked with determining the area of their function. Each function was slightly different; some functions had entirely positive outputs, some had entirely negative outputs, and some had a mix of positive and negative outputs. This was deliberate, as it gave students an opportunity to understand the area of a function as respective to the *x*-axis no matter how the function was arranged. Each group had to use all five shapes to approximate the area of their function. Using the measurements found in the Launch, students arranged their shapes to gather the best approximation possible and recorded the result in the data table. It is important to note that groups were instructed to use only one shape at a time; they could not mix triangles and rectangles for instance. This is because formal integration uses the function to generate the dimensions of shapes used to calculate the area, so only one type of shape is systematically calculated. Once the group obtained a numerical approximation for the area of their function, they turned their attention to the shape itself. Students recorded their thoughts about the effectiveness of each shape’s approximation, as well as what changes they would make to the shape to improve the approximation. For example, students noted that the circle was not ideal for approximating functions with defined peaks due to its roundness, and suggested shrinking the circle to fit better. This process was completed for all five shapes with each group's function. The purpose of the Explore phase was to have students actively working with a function and thinking intuitively about ways to improve their approximations.

Finally, the class reconvened for the Summarize phase. Each group was asked to share what their favorite and least favorite shape was for approximating their function and justify their response. Once all groups had shared, the class discussed what improvements could be made to the shapes to gather a better approximation of the function's area. The goal of the Summarize phase was to find common themes across all of the groups and extend these themes to conjectures about integration. Specifically, the class tried to ascertain which shape was the best for approximation, what qualities that shape had that led it to be the best, what could be done to the shape to improve the approximation even further, and how many shapes would yield the best approximation. Through a rich discussion, the Summarize phase was specifically designed to capture key aspects of the lesson that could be extended to the main conceptual understanding of integration.

Differentiation Lesson

The differentiation lesson was taught about one month after the integration lesson and focused on the conceptual understanding of many seemingly procedural derivative rules. Namely, students explored the power rule and product rule for calculating derivatives in a hands-on geometric manner. It is important to note that both classes had finished their units on differentiation at the time of this lesson. Both lessons were only one day in length, so students had to take a brief hiatus from their current unit to return to differentiation. The switch in content was deemed useful because students would continue to need differentiation skills in future units, and so the lesson could serve as a review activity for these concepts.

Appendix E contains the full differentiation lesson plan along with both versions of the derivative activity. The original activity was modified following the Davenport lesson based on feedback from the Davenport students and observations of the lesson, and so Appendix E contains both documents. According to the current literature, students tend to grasp the procedural notion of differentiation but struggle when grappling with conceptual understandings of the derivative (Orton, 1983). Additionally, instruction on differentiation often results in rules that are committed to memory rather than concepts that are explored and interpreted (Habre & Abboud, 2006). Consequently, the differentiation lesson focused on the underlying geometric intuition of both the power and product rules for derivatives.

In the Launch phase, students worked in pairs to create a definition for both the power rule and product rule. Many students shared how to do both of these rules but did not capture the true definition. Once each pair shared their definition, a class discussion ensued about how the rules could actually be defined without a procedure and what concepts might underlie such a potential definition. The pairs were then given the guided activity in Appendix E and the set of base-10 manipulatives shown in Figure 2.

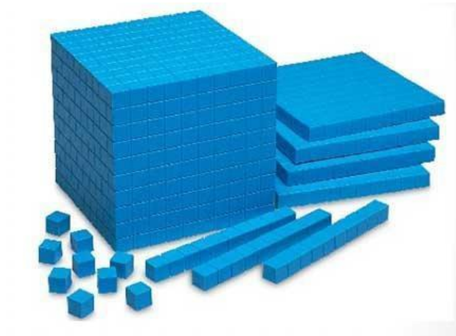

Figure 2. Base-10 Manipulatives

During the Explore phase, students worked to complete the guided activity using the manipulatives. The manipulatives were helpful in modeling functions geometrically; for instance, students used the square blocks to represent x^2 and the cube to represent x^3. In some cases, the class reconvened to address common challenges or misconceptions among the pairs, but for the most part, students worked to build their own intuition and understanding. This stage of the lesson was designed so that students could productively struggle in trying to visually understand derivatives. Difficulties were anticipated given the class's prior procedural knowledge of differentiation, but the goal was to connect this knowledge to a new and intuitive way of thinking about the same concepts. Student questions were generally addressed within the pairs, although sometimes groups mingled and discussed their conjectures and findings.

Finally, the class regrouped for the Summarize phase of the lesson. In this stage of the lesson, students discussed what exactly was discovered throughout the activity and how it related to their prior derivative knowledge. Specifically, students talked about how visually seeing where the power rule came from was helpful in understanding its origin, but that the rule itself was more efficient in practical derivative problems. Overall, this section of the lesson, albeit, shorter than the Summarize phase of the integration lesson, highlighted the key findings from the activity and rephrased them in the context of differentiation concepts as a whole.

Data Collection

Data was collected to answer each of the research questions. Table 2 lists the data collected by research question, and the subsequent sections offer more detail about this data collection.

Table 2

Data Collected by Research Question

Research Question	Collected Data
1: What types of engagement do students display throughout two high school calculus lesson?	Classroom video MP3 recordings of each student group Written exit tickets following each lesson Focus group audio Transcriptions of all video and audio data
2: How do students perceive the LES method of instruction, both in terms of mathematical learning and engagement?	Classroom video MP3 recordings of each student group Written exit tickets following each lesson Focus group audio Transcriptions of all video and audio data
3: How do teachers perceive the LES method of instruction, both in terms of mathematical learning and engagement?	Google Form Audio from student group conversations

Research Question 1

Video, audio, and written data were collected throughout this project. According to the teaching experiment methodology, each lesson should be documented in some manner to record the events that transpire throughout the duration of the class (Steffe & Thompson, 2000). Each of the two LES lessons were recorded from a stationary tripod camera in the corner of the classroom. Additionally, I wore a lapel microphone to record audio throughout the lesson. Each group of students had a small MP3 audio recorder that captured any discussion not recorded by the camera or my microphone. Finally, each student completed an exit ticket at the end of each lesson. The exit ticket, which can be found in Appendix F, allowed students an opportunity to document their reactions to the lesson. It also included a brief content check to assess how well students retained and understood the main concepts of the class period. Collectively, the video, audio, and written feedback provide a detailed record of each lesson that can be used for analysis.

It is noteworthy that the order in which the lessons were taught reversed with subject. The integration lesson was first taught at Shepherd and then at Davenport, while the differentiation lesson was first taught at Davenport and then at Shepherd. Reversing the order was deliberate to be consistent, as changes to the lessons could be made following the first teaching.

The other major methodological choice made was the inclusion of focus group interviews. A few days after each lesson, a random sample of five students was chosen from the Shepherd class to share feedback on the effectiveness of LES instruction. In the Davenport classes, the focus group discussions occurred organically following the conclusion of each lesson and therefore it was decided to not repeat the process. Many of the questions stemmed from the protocol in Appendix G, but due to the nature of the focus group discussion, not all questions that were asked are found in the appendix. Because the majority of this research is exploratory, the focus group protocol is effective in obtaining a wide and honest selection of student responses (Vaughn, Schumm, & Sinagub, 1996). The focus group protocol was centered on how students liked or disliked the LES methodology, as well as how it compared with other forms of mathematical instruction. Students were also given the opportunity to discuss how they would improve the LES lesson during the focus group, allowing them an active voice in the learning process.

All of the data described relates to the first research question. Specifically, the goal was to learn what types of engagement students display during an LES lesson. Having video and audio data of class discussions, individual group discussions, and focus groups allowed for the analysis of engagement themes. For instance, such themes included how often students conversed with one another and how much of the conversation was related to the lesson.

Research Question 2

The second research question asks how students perceive LES instruction in terms of learning and engagement. Given the fact that the second research question is focused on student perceptions of LES instruction, it was imperative to ensure that all analysis came directly from student data.

Like the first research question, the focus groups and exit tickets were a key source of data for the second research question. During the focus groups, students were asked to comment on their lesson engagement and learning. Specifically, they provided feedback on which parts of the lesson were the most engaging and helpful for learning, along with any changes they would make to improve engagement and learning. Similarly, the exit ticket asked if the lesson was engaging and how students would change the lesson if it was not engaging, as well as what the class learned about during the period. Using direct quotations from these exit tickets, as well overall themes from the focus group conversations, student perceptions of the lesson were gathered and available for qualitative analysis. Collective, this data provided an answer to the second research question.

Research Question 3

The third and final research question is concerned with how classroom teachers perceive the LES lesson both with regard to student engagement and student learning. While some teacher feedback is included in the class transcriptions of each lesson, both teachers were also asked to complete a feedback survey once both lessons were completed. The form, outlined in Appendix H, was sent via Google Forms to both teachers and responses were collected automatically. Their responses were useful in determining reactions and interpretations of the LES lessons. Additionally, there was at least one instance of teacher feedback during the class period in all four lessons. These instances also provided insight into how each teacher perceived the instruction.

Although the survey responses and classroom comments are relatively brief, they are incredibly revealing and effective for answering the third research question. Since there were only two classroom teachers involved in the project, no specific methodology was used to analyze the results. Instead, direct quotations and interpretation will be used to analyze this section of data.

Data Analysis

In the following section, the theoretical background of the coding process used for analysis, as well as specific developments of the codebook used in this project will be discussed. Following this discussion, the data that were used to answer each research question and how this data is outlined in the results will also be explained.

Coding Process

To analyze the data collected in the project, both an ongoing and retrospective analysis were performed. To do so, data-driven codes were developed based on student responses to LES instruction. This method is based in literature on analyzing interview and transcription data (Ryan & Bernard, 2003) and allows for the analyst to develop and update codes as the data is examined. Using ATLAS.ti software, transcripts of the audio and video data were openly coded to explore the data and search for themes. Specifically, open coding is the process by which data is systematically investigated to discover recurrent themes (Corbin & Strauss, 2008). For this particular project, the video and audio data collected from each lesson was transcribed before openly coded. Themes were discovered between lessons which continued to drive the types of codes that were used and allowed for a deeper analysis.

Each of the preceding components of the coding process also relate to Creswell's (2007) constant comparative method of data analysis. In this methodology, information is taken from data collection and then compared to emerging categories of the data (p. 64). For this specific project, the video and audio data produced two types of categories. The first is related to the coding itself, as the data was coded according to the types of comments that were shared throughout the lesson. The second type of category is directly related to the research questions. As the data was analyzed, it became clear which research question was addressed by different results. In turn, the data was categorized by research question, and the categories continued to develop as more results became clear. Collectively, the constant comparative method of analysis helped progress the analysis process and relate new information to preexisting data themes.

The Development of the Codebook

Prior to coding, the lesson transcripts were examined to determine the appropriate coding structure. Initially, the documents were to be coded freely, with codes being developed while reading the transcript. However, the content included in the transcripts related to Smith and Stein's (2011) *5 Practices* framework previously discussed in chapter two, and so it was decided that using this framework would be helpful in predetermining several codes. Recall that this framework is used by teachers to develop effective question types. Five of Smith and Stein's question types were applied to my questions, and Table 3 includes these specific questions and relationships.

Table 3

Use of 5 Practices (5P) Questions in the Codebook

5P Question Type	5P Definition	Coding Definition	Coding Example
Gathering Information	Requires immediate answer, rehearses known facts/procedures	Used to invoke a quick and straightforward answer from students. They often focus on the structure of the lesson or current thought process, and are usually asked to invoke quick recall of factual information	"Are we kind of stuck"
Probing Thinking	Asks student to articulate, elaborate, or clarify ideas	Used when talking with students either individually or in groups. The purpose is to the student(s) expand on current thinking. This is done to either aid my understanding of student thoughts and/or to subtly introduce addition things for the student(s) to think about	*Student:* "What about the area of a function" *Me:* "Right, so what does that mean to you?"
Generating Discussion	Solicits contributions from other members of the class	Used when I want to invoke a fuller group discussion about a concept or previously raised idea. May be with the whole class or within individual working groups.	(*to class*) "Do we agree or disagree with that?"

Table 3 (continued)

5P Question Type	5P Definition	Coding Definition	Coding Example
Exploring Mathematical Meanings and Relationships	Points to underlying mathematical relationships and meanings. Makes links between mathematical ideas and representations	Used when I explicitly structure my questions to explore specific mathematical concepts when talking with a student(s). I use what students say or are working on to form these questions. I also use them to relate student thinking to deeper mathematical concepts or reasoning	"And you made a new square out like this. Okay, so if I make these a little bit longer by some *dx*, what is the area of each of those new pieces that you added?"
Extending Thinking*	Extends the situation under discussion to other situations where similar ideas may be used	Used when I explicitly structure my questions to explore specific mathematical concepts when talking with students. I use what students say or are working on to form these questions, and also use them to extend student thinking to deeper mathematical concepts or reason. This code is a step above the last. Thus, if a quote can be coded with both, only this one will be used.	*Student:* "But we can't fill in all the tiny shapes either. Like if you have a really small shape, that's going to be hard to get onto the board." *Me:* "Right, so maybe if we take out the human part of it. What if we could do this automatically where we didn't have to physically draw it?"

*This code was introduced in Round 2 of the codebook development.

The last step in the coding process was to establish reliability of the codebook. The focus groups were coded for overall themes and therefore not subject to measures of reliability. To reach reliability, a portion of both differentiation lessons was chosen, including quotations from the full lesson transcript and individual group transcripts, and coded individually by my advisor and myself. My advisor and I compared our coding choices for each line of the transcriptions, and counted the number of identical and non-identical choices. This process was repeated three times until the number of identical choices was at least 80%, as is routine for research involving inter-rater reliability (Bernard, 2017). Table 4 outlines the progression of the codebook, while Appendix I contains the complete final codebook used for analysis. Critical codebook changes are explained in the following paragraphs.

In the inaugural round of coding, the data were analyzed under four main categories: Lesson Feedback, Logistics, Personal Statements and Student Statements. The Personal Statements category contained all of my questions and comments throughout the lesson, and the Student Statements category contained all student questions and comments throughout the lesson. Through open coding and constant comparison, 17 codes emerged.

For the second round of coding, my advisor and I applied these 17 codes to sections of the Shepherd differentiation lesson. These sections included components from both the full class transcript and individual group transcripts. During our reliability discussion, it quickly became apparent that the codebook needed more structure to be effective and accurate. Consequently, five categories were redefined and extended to encompass 24 codes. The biggest change in this round of coding was the separation of comments and questions. Originally, student comments and questions were combined into one category, and my personal comments and questions were combined into another category. However, the new set of categories included Student Comment Types, Student Question Types, Personal Comment Types, and Personal Question Types.

Table 4

Codebook Development

Round 1 Coding	Round 2 Coding	Round 3 Coding
Lesson Feedback • Classroom Teacher Feedback or Explanation • Negative Lesson Feedback • Positive Lesson Feedback **Logistics** • Focus Group Directions • LES Explanation • Lesson Directions **Personal Statements** • Affective • Gathering Information • Probing Thinking • Exploring Math Meanings • Generating Discussion **Student Statements** • Lesson Confusion/Questions • Gathering Information • Building Connections • Mathematical Inquiry (comment) • Mathematical Inquiry (question) • Inserting Technology	NEW **Feedback/Logistics** • Future Lesson Think About **Personal Comment Types (PCT)** • General Response PCT • Rephrasing and Extending PCT • LES Explanation • Lesson Directions • Affective PCT **Personal Question Types (PQT)** • Gathering Info PQT • Probing Thinking PQT • Rephrasing and Extending PQT • Generating Discussion PQT • Affective PQT **Student Comment Types (SCT)** • General Response SCT • Active Thinking SCT • Mathematical Inquiry SCT • Building Connections SCT • Lesson Confusion SCT • Off-Topic SCT	NEW **Feedback/Logistics** • Off-Topic **Personal Comment Types (PCT)** • Exploring Math Meanings Relationships PCT **Personal Question Types (PQT)** • Exploring Math Meanings and Relationships PQT MERGED **Personal Comment Types (PCT)** • General Response PCT • Rephrasing and Extending PCT • LES Explanation • Lesson Directions • Affective PCT **Personal Question Types (PQT)** • Gathering Info PQT • Probing Thinking PQT • Rephrasing and Extending PQT • Generating Discussion PQT • Affective PQT

Table 4 (continued)

Round 1 Coding	Round 2 Coding	Round 3 Coding
	Student Question Types (SQT) ● Gathering Info SQT ● Active Thinking SQT ● Mathematical Inquiry SQT ● Lesson Confusion SQT MERGED ● Classroom Teacher Feedback/Explanations ● Negative Lesson Feedback ● Positive Lesson Feedback REMOVED ● Inserting Technology ● Focus Group Directions ● Exploring Math Meanings	**Student Comment Types (SCT)** ● General Response SCT ● Active Thinking SCT ● Mathematical Inquiry SCT ● Building Connections SCT ● Lesson Confusion SCT **Student Question Types (SQT)** ● Gathering Info SQT ● Active Thinking SQT ● Mathematical Inquiry SQT ● Lesson Confusion SQT **Feedback/Logistics** ● Future Lesson Think About ● Classroom Teacher Feedback and Explanation ● Negative Lesson Feedback ● Positive Lesson Feedback REMOVED ● Off-Topic SCT

This distinction allowed for a deeper analysis of what types of questions students were asking, what levels of student thought my questions were invoking, and what levels of student thought occurred during student discussions.

In the final stage of codebook development, only three new codes were added, bringing the total number of codes to 26. It was in this stage that definitions of codes were finalized and applied in a specific manner. For instance, the Off-Topic code was introduced to use whenever a quotation in the data was irrelevant to analysis. Rather than use an existing code that might have produced erroneous results, it was decided that such quotations would be ignored to avoid unintentionally skewing the data.

Additionally, the Exploring Math Meanings/Relationships code was reintroduced.

Another key aspect of this stage was formally defining what the Mathematical Inquiry code meant. Prior to this round of reliability, defining student mathematical inquiry was challenging. After discussions, it was determined that, if a student made a comment or asked a question directly related to the central foci of the lesson, it would be coded as Mathematical Inquiry. For instance, a student in the integration lesson noted that “with the smaller shapes, we had more certainty with the estimation.” Noticing the benefits of using smaller shapes for area approximations is a key focal point in the integration lesson, and thus the student comment was coded as Mathematical Inquiry.

Table 1 shows the pairing of the Gathering Information question type with the General Response student comment type. Since Gathering Information questions are used to recall factual information or procedures, there was not a significant level of cognitive demand placed on the student. Consequently, the corresponding student comments were general, often just stating the fact or procedure about which I asked. Since this question and response pair was largely recall, it is considered the lowest level of cognitive demand. It is worth noting however that, even though it is the least amount of cognitive demand, this question and response pair was important in the lessons. For instance, such questions were needed to help students who were struggling to get started on a task, as it was a way to remind them of something they already knew to begin considering a topic in a deeper manner. Such was the case in the differentiation lesson, as students were initially asked to define the power rule and product rule before they used the manipulatives to explore these concepts in more detail.

The middle level of cognitive demand involves the Probing Thinking question type and Active Thinking student response types. As noted in Table 3, Probing Questions were used to elicit more justified and developed student responses. These questions took a more concrete idea or inquiry and delved deeper into the meaning behind the concept or claim. As a result, students achieved a higher cognitive demand in answering such questions. Probing Questions and Active Thinking responses are matched, as they both involve a level of cognitive demand that is higher than a basic response but not as sophisticated as Mathematical Inquiry.

The final level of cognitive demand involves two question types and one student response. In response to the Exploring Math Meanings and Extending Thinking question types, students often demonstrated levels of Mathematical Inquiry. Table 3 shows that both question types are meant to develop an understanding of underlying mathematical structure, and connect this structure to more holistic and complex mathematical ideas. For instance, in the differentiation lesson, students first discussed geometric areas of the manipulatives. Once they established this concept, they used it to explore rates of change, a much deeper idea and the central focus of the lesson. In general, because students had to think the most about the concepts included in the Exploring Math Meanings and Extending Thinking questions, and these concepts were the foci of the lesson, this personal question and student response pairing has the highest cognitive demand.

Data Analysis for Research Question 1

The first research question asks what types of engagement students exhibit during the LES lessons. In order to answer this question, code frequencies were analyzed with respect to one another. The percentage of each code respective to the total number of codes from that code group and total number of codes overall was determined. From there, specific themes were explored that yielded insight into the research question. Specifically, it was of interest how often student discussion and deeper levels of thought generated other students' thinking at the same level. Thus, one result is how often a student exhibited Active Thinking or Mathematical Inquiry following another student who also exhibited Active Thinking or Mathematical Inquiry. This analysis was important because LES instruction is meant to be communal and student-based.

An additional result is the frequency of when the Exploring Mathematical Meanings or Rephrasing and Extending questions and comments elicited an Active Thinking or Mathematical Inquiry student response. In other words, did the highest cognitive demand questions invoke the highest level of students' responses?

Finally, the data were analyzed for all instances of Probing Questions and compared to what followed these inquiries. Because the Probing Questions were meant to be an intermediate question type, it was appropriate to determine how students responded. Additionally, this data helped answer the first research question because it provided insight into how students used an intermediate question in their discussion, and what types of engagement followed such a question. For instance, some students used Probing Questions as a bridge to more Active Thinking, while other students asked additional questions regarding the content included in my Probing Questions.

Data Analysis for Research Question 2

The second research question asks how students perceive LES lessons, both in terms of engagement and mathematical learning. To answer this question, the method of constant comparison was used to look for major themes across the data from the focus groups, exit tickets of each class, and student comments about the lesson during teaching. In addition, qualitative analysis was employed to collect direct quotations that exemplified emerging themes.

Data Analysis for Research Question 3

The final research question is concerned with how the classroom teachers perceived LES instruction, both in terms of student engagement and student learning. This question is identical to the second research question, except all perceptions came from the participating classroom teachers. Because there were only two teachers, responses to the online form and teacher feedback during the lesson were summarized and analyzed for overall themes. Any direct quotations from the lesson transcripts are also included in the summary as supplementary information, and the summaries are separated by teacher.

CHAPTER 5: RESULTS

In this chapter, the data outlined in chapter four is supplied and separated by research question. Practical changes to both the differentiation and integration lesson based on the findings are then explored.

Results by Research Question

Research Question 1

The first result is the total percentage of the codes in each category of the codebook. Table 5 contains the distribution of all project codes separated by code group.

Table 5

Percentage of Project Codes by Code Group

Code Group	Percent of All Project Codes
Personal Comment Types	14.5%
Personal Question Types	15.1%
Student Comment Types	42%
Student Question Types	13.4%
Feedback and Logistics	14.7%

From Table 5, the majority of codes in the project were student comments. All other code groups had similar percentages. Collectively, student quotations composed 55.4% of all codes and personal quotations composed 29.6% of all codes.

Table 6 includes the overall code counts for the Personal Comment Type category. The percentage of each code respective to the total number of codes from that group and total number of codes overall is also given.

Table 6

Personal Comment Types Code Distribution

Code	Percent of Code Group	Percent of Total Codes
General Response PCT	101/368 = 27.5%	101/2542 = 4.0%
Lesson Directions	109/368 = 29.6%	109/2542 = 4.3%
LES Explanation	22/368 = 6.0%	22/2542 = 0.9%
Affective PCT	59/368 = 16%	59/2542 = 2.3%
Exploring Math Meanings and Relationships	39/368 = 10.6%	39/2542 = 1.5%
Rephrasing and Extending	39/368 = 10.6%	39/2542 = 1.5%

From Table 6, 57.1% of my personal comments were either general responses or lesson directions. About 21% of my comments were cognitively demanding, and 14.5% of all codes in the project were my comments.

Table 7 includes the overall code counts for the Personal Question Type category. The percentage of each code respective to the total number of codes from that group and total number of codes overall is also given.

Table 7

Personal Question Types Code Distribution

Code	Percent of Code Group	Percent of Total Codes
Affective PQT	6/387 = 1.6%	6/2542 = 0.2%
Gathering Information	133/387 = 34.4%	133/2542 = 5.2%
Probing Thinking	92/387 = 23.8%	92/2542 = 3.6%
Exploring Math Meanings & Relationships	90/387 = 23.3%	90/2542 = 3.5%
Rephrasing and Extending	46/387 = 11.9%	46/2542 = 1.8%
Generating Discussion	20/387 = 5.2%	20/2542 = 0.8%

From Table 7, about 64% of my personal questions at least reached the intermediate level of cognitive demand, and 40.4% of my questions reached the highest level. About 15% of the codes in the project were personal question types as well.

Table 8 includes the overall code counts for the Student Comment Types category. The percentage of each code respective to the total number of codes from that group and total number of codes overall is also given.

Table 8

Student Comment Types Code Distribution

Code	Percent of Code Group	Percent of Total Codes
General Response	503/1066 = 47.2%	503/2542 = 19.8%
Lesson Confusion	39/1066 = 3.7%	39/2542 = 1.5%
Active Thinking	363/1066 = 34.1%	363/2542 = 14.3%
Mathematical Inquiry	126/1066 = 11.9%	126/2542 = 5.0%
Building Connections	35/1066 = 3.3%	35/2542 = 1.4%

From Table 8, nearly 50% of student comments reached the highest level of cognitive demand, and the other 50% of comments were general responses. Only about 4% were associated with lesson confusion. Overall, nearly 20% of all codes in the project were intermediate or high cognitive demand student comments.

Table 9 includes the overall code counts for the Student Question Types category. The percentage of each code respective to the total number of codes from that group and total number of codes overall is also given.

Table 9

Student Question Types Code Distribution

Code	Percent of Code Group	Percent of Total Codes
Lesson Confusion	24/346 = 6.9%	24/2542 = 0.9%
Gathering Information	165/346 = 47.8%	165/2542 = 6.3%
Active Thinking	131/346 = 37.9%	131/2542 = 5.2%
Mathematical Inquiry	26/346 = 7.5%	26/2542 = 1.0%

From Table 9, 45.4% of student questions at least reached the intermediate level of cognitive demand. Only a small percentage of student questions were actually about lesson confusion.

Now that an overall distribution of codes has been established, consider the specific coding patterns in each lesson. Table 10 separates the data by lesson, and reports the percentage of student Active Thinking or Mathematical Inquiry that was immediately followed by another student's Active Thinking or Mathematical Inquiry. Essentially, this table shows how often higher levels of student cognitive demand led to equally high cognitive responses by other students.

Table 10

Ongoing Student Active Thinking and Mathematical Inquiry Discussions

Lesson	Active Thinking/Mathematical Inquiry Following Active Thinking or Mathematical Inquiry
Davenport Differentiation	64/185 = 35%
Davenport Integration	71/192 = 37%
Shepherd Differentiation	77/227 = 34%
Shepherd Integration	6/42 = 14%

Note that in Table 10, there were some technical issues with the audio recorders for each individual Shepherd integration group. Given the other lesson data, it is likely that there were more instances of Active Thinking or Mathematical Inquiry following other Active Thinking or Mathematical Inquiry, but since the recorders malfunctioned, it is not certain. All of the other lessons have relatively equal percentages of Active Thinking and Inquiry followed by Active Thinking and Inquiry, and these percentages are between 34% and 37%.

Because one of the goals of LES instruction is to discover underlying mathematical concepts and principles, and since higher level questioning types are specifically meant to accomplish this goal, it was useful to determine how often these deeper questions generated deeper student responses. Table 11 again separates the data by lesson and reports the percentage of Exploring Math Meanings and Rephrasing and Extending question types that were immediately followed by a student's Active Thinking, Mathematical Inquiry, or Building Connections response. The final column is all other codes that followed the Exploring Math Meanings and Rephrasing and Extending questions.

Table 11

High Cognitive Questions followed by High Cognitive Responses

Lesson	Total Exploring Math Meanings & Rephrase and Extending PCT/PQT	Single Active Thinking, Mathematical Inquiry, Building Connections following Exploring Math Meanings, Rephrase and Extending PCT or PQT	Multiple Active Thinking, Mathematical Inquiry, Building Connections following Exploring Math Meanings, Rephrase and Extending PCT or PQT	Other Codes following Explore Math Meanings or Rephrase and Extend PCT or PQT
Davenport Differentiation	66	25/66 = 38%	6/66 = 9%	35/66 = 53%
Davenport Integration	33	15/33 = 46%	11/33 = 33%	7/33 = 21%
Shepherd Differentiation	90	44/90 = 49%	10/90 = 11%	36/90 = 40%
Shepherd Integration	21	17/21 = 81%	2/21 = 9.5%	2/21 = 9.5%

Table 11 indicates that the majority of my highest cognitive demand questions in each lesson resulted in an intermediate or high cognitive demand student response in every lesson except the Davenport differentiation lesson. This lesson was nearly the majority at 47%.

In an effort to ascertain how effective Probing Questions were in generating at least intermediate levels of student cognition, Table 12 reports what codes followed the Probing Questions in each lesson. These results are given as percentages of all Probing Questions by lesson.

Table 12

Codes that followed Probing Questions

Lesson	General Response SCT	Gathering Information SCT	Active Thinking SCT/SQT	Mathematical Inquiry SCT/SQT	Building Connections SCT/SQT	Negative Lesson Feedback
Davenport Differentiation	3/28 = 11%	1/28 = 3.6%	19/28 = 67.9%	2/28 = 7.1%	2/28 = 7.1%	1/28 = 3.6%
Davenport Integration	6/24 = 25%	2/24 = 8.3%	7/24 = 29.2%	9/24 = 37.5%	0	0
Shepherd Differentiation	5/37= 13.5%	3/37 = 8.1%	23/37 = 62.2%	6/37 = 16.2%	0	0
Shepherd Integration	3/14 = 21.4%	0	1/14 = 7.1%	10/14 = 71.4%	0	0

According to Table 12, the majority of codes that followed Probing Questions were either Active thinking or Mathematical Inquiry. In the four lessons, these were 75%, 67%, 78%, and 79%, respectively. The second highest response was either a General Student Response or a Gathering Information student question.

Research Question 2

The results for the second research question are partially qualitative and partially quantitative. Table 13 displays the overall percentages of each code in the Feedback and Logistics code group. This table also includes the percentages of each code with respect to the entire project.

Table 13

Feedback and Logistics Code Distribution

Code	Percent of Code Group	Percent of Total Codes
Future Lesson Think About	52/375 = 13.9%	52/2542 = 2.0%
Positive Lesson Feedback	47/375 = 12.5%	47/2542 = 1.8%
Negative Lesson Feedback	17/375 = 4.5%	17/2542 = 0.7%
Off-Topic	193/375 = 51.5%	193/2542 = 7.6%
Classroom Teacher Feedback/Explanation	66/375 = 17.6%	66/2542 = 2.6%

From Table 13, about half of the Feedback and Logistics codes were Off-Topic and therefore of no relevance to analysis. These codes made up about 8% of the total codes in the project. Only 3% of codes in the project came from classroom teachers, and the percent of positive feedback is nearly triple that of negative feedback.

Next, student feedback about their engagement and learning is summarized. All information in the summary came explicitly from the exit tickets and focus groups. For more detail about these results, see Appendix J. With regard to engagement, the majority of students agreed that both lessons were highly engaging and interactive. Several students reported that they were engaged because they “like hands-on learning” and because it was different than other classes since “there aren’t a lot of classes that make you do hands-on things and get up with other people.” Students reported that the integration lesson was more engaging than the differentiation lesson because the differentiation lesson “needed to be more understandable.” Finally, several students noted that both lessons were more engaging than traditional instruction. For example, one students said “I liked it because we were learning something new but not in a boring way” while another said that “it was a fun activity that was more entertaining than sitting at a desk staring at a piece of paper." Overall, student feedback on lesson engagement was positive, but there were some areas that could be improved.

Student feedback regarding learning the essential concepts of each lesson was more varied than engagement feedback. For the integration lesson, many students felt that the visual aspects of the lesson helped them learn. For instance, one student noted that "I liked being able to use the shapes. We weren't just given a formula and told to plug numbers in" while another said that "it wasn't a forced application" of the concepts. However, some students noted that they "learn better with formulas, numbers, and examples" and that the approximations made it difficult because they "like actual answers."

The differentiation lesson had similar patterns. Students again reported that the visuals helped understand the concepts. Additionally, one student reported that it was helpful for learning because "it was a challenge to think about what derivatives actually mean." Many students felt that they could not fully learn the concepts from the lesson because they either "already had an understanding" or because they were confused. Students also found it difficult to use the manipulatives because they "wanted to associate them with a number rather than the concept itself." Between both lessons, student feedback on learning was certainly more scattered. However, there was a decent amount of positive feedback as well as constructive feedback for future lesson use.

Finally, the feedback generated entirely within each lesson is reported. All data are quotes from the lesson transcripts that were coded with Positive Lesson Feedback, Negative Lesson Feedback, or Future Lesson Think About. A full report of this data is found in Appendix K.

With regard to positive feedback, many student comments expressed excitement about the lesson or amazement about a mathematical result. For instance, one student said “I just had a mathematical breakthrough” while several others responded to major lesson results with an astonished “What?!” or “Whoa.” Several students also commented on how it was “nice knowing that the math we’re doing actually means something.”

In terms of negative feedback, one student commented “give me an equation, I’ll do the math and figure it out. I’m never going to need this in life.” Other instances of negative feedback included frustration or giving up, as well as a lack of care. For instance, one student noted that “after today we won’t have to remember it [the lesson]” while others said that they should “give up” or “quit.”

Research Question 3

First, Ray’s (the Shepherd teacher) feedback is summarized. Prior to this project, Ray did not have any knowledge of LES instruction. He was skeptical of the method following the lessons, arguing that “most students want you to tell them what to know and how to complete a process.” Ray also said that students “get frustrated if you ask them to do something you haven’t previously walked them through.” In terms of engagement, he noted that LES instruction can be effective with good preparation.

However, lessons that "do not explain expectations may lead to a lot of frustration and wasted time." In terms of student learning, Ray "absolutely" thought the LES lessons were effective. He also noted that the "students in the class that Nate completed an integration activity with connect definite integrals with area much better than the class he did not teach." For future lessons, Ray shared that ongoing LES instruction may be challenging and time consuming for the teacher. It may better serve as a "great compliment to traditional teaching, as students may get frustrated with constantly 'discovering' a concept." Finally, Ray expressed concern with absent students, as it would be difficult to make up an LES lesson as opposed to copying notes.

Next, Troy's (the Davenport teacher) feedback is summarized. Troy also did not have any prior knowledge of the LES method, but said he has seen similar strategies in his experience. His biggest concern following the lesson was the amount of time it would take to structure a curriculum as an LES environment. He felt that it "would be hard to get through the entire curriculum doing it with LES method." In terms of engagement, Troy noted that students were "into the [integration] lesson very much," but that the differentiation lesson "seemed to frustrate them more." However, he also shared that this is "not always a bad thing." With regard to student learning, Troy thought that LES instruction helped students "because they now had a visual and hands-on idea of the concepts." For future lessons, he classified LES as a "great theory-based idea, but not very practical because of our 42 minute class periods." He was unable to see how to complete an entire curriculum using LES instruction. Finally, despite his inclination that LES would enhance students' critical thinking abilities, Troy figured that many students would grow tired of the method by the third or fourth lesson.

Comparatively speaking, Troy and Ray both expressed concerns with the longevity of LES instruction in high school classes. They felt the structure of the lessons and amount of preparation needed for each lesson would be too much to sustain a full curriculum. Both Ray and Troy felt that LES instruction helped students learn, although they differed on specific concepts. Ray felt that students learned the most about connection between definite integrals and area under curves, while Troy felt that LES instruction best helped students' critical thinking abilities.

Practical Changes to Each Lesson

Before discussing the results of each research question and the theoretical implications this study has for teaching and learning calculus, an overview of changes to each lesson must be considered. These adaptations are considered a practical result of the data analysis because, although the purpose of this study was to determine the effectiveness of LES instruction in teaching calculus, two rich calculus lessons were developed and are available for future use. These changes are in direct response to both the data collected and personal reflection following each lesson.

Integration Lesson Changes

Between both lessons, the integration lesson is certainly the more complete design, as there are only a few changes that need to be made to this lesson before using it again. The first is allotting at least two class periods for the activity. Although the logistics of this project restricted the amount of class time spent on each lesson, more time should be devoted to students' exploration and discussion of their early understandings of the definite integral. This was most evident in the difference between the Summarize phase of the lesson at Shepherd and the Summarize phase at Davenport. Although both discussions yielded good insight and student thought, the Davenport students experienced a richer debate that involved multiple conjectures about improving approximations, justification for these improvements, and concerns with calculating the area of infinitely many shapes. The Shepherd students briefly touched on each of these points, but because the entire lesson had to be completed in one class period, there was not enough time to fully develop ideas. The Davenport lesson spanned over two class periods and therefore allowed the class to methodically conceptualize and discuss the findings of the lesson.

Further, an explicit distinction between a large shape and a small shape should be made. Although students were given a parallelogram with an area of 600 square centimeters (the largest of the five shapes) and a triangle with an area of 200 square centimeters (the smallest of the five shapes), not all students compared these specific shapes in their analysis. Rather, students often noted that they liked the triangle because it was small, and not necessarily because it was the smallest. The size of the parallelogram should be exaggerated to invoke a more direct comparison.

The final change is with regard to the actual area approximations that each group made. Specific function graphs were built to have interesting properties, including sharp peaks, rounded curves, and positive and/or negative sections. While the peaks and curves of the graphs were effective in helping students thinking about which shapes were best for approximation, the groups neglected to use signed measurements when calculating their approximations. This essentially negated the rationale for creating the graphs in this manner, even though the rationale was explained before beginning the Explore phase. During the lesson, it was ultimately decided that an understanding of improved approximations was more important for an introduction to integration as opposed to revisiting the signed approximation discussion. In a future lesson, students should be reminded about this discussion, thus providing substance to the deliberate function creation.

Differentiation Lesson Changes

The differentiation lesson had a wider range of student responses and feedback than the integration lesson, and is therefore more susceptible to change. Like the integration lesson, at least two class periods should be allocated for this lesson. Although the original differentiation lesson plan included a section on the power rule and another section on the product rule, none of the groups at either school got to the product rule section. Moreover, each class only had a limited amount of time at the end of the lesson to collectively discuss its findings. Rather, the class quickly summarized the activity and its implications, and therefore could not discuss the lesson as much as originally planned. Having at least two class periods would allow each group to complete and think about the entire activity while preserving time at the end of class to make deeper connections with the material.

The second change to the differentiation lesson would be its place in the overarching calculus curriculum. At the time of teaching this lesson, both classes were in the middle of an integration unit and had already completed their differentiation unit. While this activity still served as a nice review of some essential differentiation concepts, many students reported that they would rather have completed it when they first learned about the power rule and product rule. One student in particular noted that it was difficult to see the purpose of a more in-depth activity because the class already knew how to use the derivative rules. Rather, this activity would fit in conjunction with other power and product rule lessons to help students understand the concepts behind each rule, as well as why the rules are used in practical situations.

The third change to this lesson involves the language and wording of the guided activity in Appendix E. Both the original activity and the revised version appear in this appendix, but further changes are still necessary. In the focus groups for this lesson, students reported that the clarity of the activity was a large barrier to understanding its concepts and purpose. Specifically, it seemed that students were uncomfortable using Leibniz notation to describe changing quantities. For instance, the activity asks students to consider a small change in the length of x^2 by representing it as dx. Many groups struggled with this question, and in turn could not use such quantities for later algebraic manipulation. Students should first be familiar with Leibniz notation prior to teaching this lesson.

The other main issue with the guided activity involved the algebraic manipulation of changing quantities. This question specifically asked students to describe how df changes proportionally to dx. The purpose of this question was to illustrate how small changes in x affect large changes in the function f. In context of the activity, this involved changing the side length of one of the square manipulatives and building a new square from these revised dimensions.

Many students were uncomfortable with the concept of proportions, and therefore needed some scaffolding to realize that df/dx was the quantity of interest. Again, this stems slightly from an unfamiliarity with Leibniz notation, but the question should be reworded as follows: Finally, try to algebraically manipulate your expression to isolate the ratio of how f changes over how x changes. (Hint: Think about which of the previous quantities represent the change in f and the change in x!).

Finally, the manipulatives themselves must be changed. Students reported that they struggled with abstract measures of length and width using the base-10 blocks shown in Figure 2. As one student pointed out, "it was confusing because I wanted to associate the blocks with numbers rather than concepts." Using physical base-10 blocks as opposed to digital base-10 blocks is more valuable because students can actually manipulate changing quantities with their own hands. Consequently, actual manipulatives should still be used rather than a digital equivalent. However, the base-10 blocks should not have notches meant for counting. In other words, the new manipulatives should have the same dimensions as base-10 blocks, but eliminate the possibility of students counting the blocks to obtain a length or width. This way, abstract quantities for length and width make sense for an unknown measurement rather than trying to fit a variable to an already known quantity.

CHAPTER 6: DISCUSSION

Interpretation of the Results

In this section, the results of each research question are discussed in context of the project. The interpretations are separated by research question.

Research Question 1

The first research question asks about what types of engagements students displayed during each lesson. From Tables 5 through 9, an explanation of the overall understanding of student lesson engagement is demonstrated, and this broad picture is helpful in beginning to answer the first research question. Of all the codes used in the project, over 50% were student comments or student questions. Less than 30% of codes were my questions and comments, while the remaining 20% came from feedback or logistics. In terms of engagement, this is an encouraging distribution of codes. The majority of conversation in each lesson was generated by the students, and many groups worked through the activities with little input from me. Instead, students were actively working to understand and discover the concepts associated with each lesson, asking strong questions to aid in this understanding, and conversing with one another to build insight. The following transcript is an example of two Shepherd students working through the differentiation activity with the codes included in brackets for reference.

> **S8**: Now that you've conquered $f(x) = x^2$, try to do a similar process with $f(x) = x^3$. [General Response SCT]
> **S9:** Hmm. [General Response SCT]
> **S8:** Oh this deals with volume. Again imagine changing x by some little amount. How would $f(x)$ change? Huh. [Active Thinking SCT]

S9: So it'd be like this? (*puts* x^2*blocks on sides of the* x^3 *block*) [Active Thinking SQT]
S8: Yeah there'd just be one on each side right? [Active Thinking SQT]
S8: And I guess we'd have to put… [Active Thinking SCT]
S9: We need these blocks there, and one on the back. [Active Thinking SCT]

Notice that these students are asking each other questions as they work with the manipulatives to model the derivative of x^3. This conversation also occurred independently of my comments or questions, again offering support that student engagement during the lesson was largely between students. Compared to traditional lectures where the majority of dialogue is teacher-based, the data suggests an opposite trend and in turn supports the notion that LES instruction is effective in promoting student engagement.

In addition to rich student dialogue, the results for the first research question indicate that a significant amount of student discourse reached the intermediate or highest levels of cognitive demand. From Table 10, more than 30% of student Active Thinking or Inquiry led to additional student Active Thinking or Inquiry, being aware of the fact that there were technological issues during the Shepherd integration lesson that limited analysis. Not only were students actively engaged and talking to one another throughout the course of the lesson, they were engaged in high level mathematical conversations. Additionally, this trend is consistent among all lessons. This suggests that the LES structure may be more responsible for high levels of discourse as opposed to the actual topics of the lessons. Although the previous transcript example also demonstrates a highly cognitive student conversation, the following transcript example provides another example from the Davenport integration lesson for comparison. These students were discussing which shapes offered the best estimation of the area of their function, and how they would improve the shapes that were not effective estimators.

> **S7:** I'd change the rectangle to a square. [Active Thinking SCT]
> **S11:** I don't know, maybe make it smaller? [Mathematical Inquiry SCT]
> **S7:** I feel like the triangle was a good size. [Active Thinking SCT]
> **S2:** Okay, parallelogram. Make it a square? [Active Thinking SQT]
> **S7:** I think this one would be better if we could make it smaller to fit better in here. [Mathematical Inquiry SCT]
> **S11:** Yeah smaller for sure. [Active Thinking SCT]

Smith and Stein (2011) found that teachers can help students with cognitively challenging tasks by asking good questions. This goal was accomplished in the project using the Exploring Math Meanings/Relationships and Rephrasing and Extending question types. Table 11 indicates how often these deepest questions elicited an intermediate or high cognitive student response. In three out of the four lessons, the majority of these questions yielded higher response. The integration and differentiation lessons differed slightly in percentages, with both integration lessons reaching 78% and 89% while the differentiation lessons reached 45% and 60%. This is likely do to the structure and clarity of the differentiation lesson. Many students reported that the questions in the derivative activity were unclear and that it was difficult to fully engage with the lesson. Consequently, there was a need to use a larger number of low cognitive demand questions to help students progress through the activity. Despite these differences, higher level questions were effective in invoking intermediate or high level student responses. The following transcript from the Davenport integration lesson is an example. Here, I am talking with two students about how to use the shapes they measured to make approximations about their functions.

> **I:** You could guess, you could make up a number. How could you make a good guess? A good approximation? Okay, what areas do I know out of these functions and out of the shapes that you have? [Exploring Math Meanings/Relationships PQT]
> **S5:** You know the areas of like the triangle and things like that. And you could possibly use it for like, that one. [Active Thinking SQT]
> **S9:** So like that one, you would find the radius of… [Active Thinking SQT]
> **I:** Well these aren't nice shapes like the ones you have. But you do have a nice circle. Alright, you do have those nice shapes that you know the area of. How could I use those shapes…? [Rephrasing and Extending PQT]
> **S9:** Oh just fill them in! [Mathematical Inquiry SCT]
> **S5:** Oh! [Mathematical Inquiry SCT]
> **S9:** Like a puzzle. [Mathematical Inquiry SCT]

Notice that this transcript includes an Exploring Math Meanings/Relationships and a Rephrasing and Extending question. Initially, S5 and S9 are working to build conjectures about how to find the area of the function without using the shapes. The Rephrasing and Extending question combines their thinking with the availability of the shapes, and ultimately results in three instances of Mathematical Inquiry.

Finally, considering the effectiveness and use of Probing Questions in engaging students with mathematical concepts is necessary. Table 12 outlines all of the responses that immediately followed one of my Probing Questions. In each lesson, at least 60% of these questions produced an Active Thinking or Mathematical Inquiry student response. Given that all four lessons demonstrate similar trends, the data suggests that Probing Questions were effective in eliciting at least an intermediate level of student cognition. Despite initially matching Probing Questions with Active Thinking student responses (see Table 1), a significant number of Probing Questions were able to help students reach the level of Mathematical Inquiry. Such results indicate that Probing Questions certainly engaged students, both with each other and with the content. The next transcript is a section of the Shepherd integration lesson when a Probing Question about improving the shapes in area approximations leads to two instances of Mathematical Inquiry and one instance of Active Thinking. Students were discussing what qualities each shape had that made it an effective or ineffective estimator. S6 for instance noted that he would not use one of the shapes because he was unable to accurately position it in the function, which is a key finding in the context of the problem. S13 highlighted a feature of trapezoids that made them better estimators than other shapes.

I: How would you make it better? [Probing Thinking PQT]
S6: I would not use it. [Mathematical Inquiry SCT]
S13: At least the trapezoid had some angles. [Mathematical Inquiry SCT]
S8: The circle was pretty good. [Active Thinking SCT]

Overall, the data for the first research question indicates that LES instruction engages students in a variety of ways. The lesson structure itself, including group work, discussions, and manipulatives encourages students to work together and generate conjectures about the material. Additionally, students have conversations about the focal mathematical concepts, and these conversations lead to additional conjectures and justification. Finally, the questions that were aimed at generating higher levels of student thought achieved their goal, and this helped students engage with the lesson and concepts as well. Collectively, these results are encouraging, as they indicate that the LES structure engages students on multiple levels, both with each other and with the mathematics. They also support the first hypothesis that data analysis will show more engagement than traditional instruction.

Research Question 2

The second research question asks how students perceive LES instruction both in terms of engagement and learning. Quantitatively, Table 13 indicates that students offered positive feedback three times more often than negative feedback. This ratio is encouraging because the data came directly from the lesson transcript and occurred naturally throughout the lessons. Additionally, this feedback was related to both engagement and learning. In many cases, positive feedback stemmed from a surprising or interesting mathematical result, or throughout the course of an interactive portion of the activity. For example, the Davenport students in the following transcript were shocked when they learned that it was possible to calculate the area of an infinite region. From a teaching standpoint, observing students get excited about the concepts they are learning about is a positive outcome. Not only does it increase student motivation to ask questions and continue learning, but it reconnects students to the innate joy of learning that is often lost in traditional forms of instruction.

> **I:** So if we make our shapes super small, and have some precise way of orienting them, and we put an infinite number of them in our function…
> **S12:** Infinity?
> **I:** Well that's exactly what you just figured out is how to get that answer, it's your…
> **S10:** But if they're infinite then they just keep going!
> **I:** Right…
> **S8:** So there's no way to calculate it.
> **I:** Ah, but there is!
> **Class:** What!? (*laughter*)
> **S5:** Shh, listen to him. Just listen!
> **I:** What you just--what you just kind of described to me and intuitively figured out is what's called an integral.

With regard to student perceptions of engagement, the focus groups and exit tickets indicate that students did feel like LES instruction was engaging. The majority of student responses identified that LES teaching is hands-on, visual, and collaborative. Students also highlighted the fact that they became the central factors of the lessons. For instance, one Davenport student said that the lesson was engaging because "we had to come up with mostly everything on our own." Another student agreed that she "liked being able to do it myself." Comparatively speaking, students in both schools reported higher levels of engagement during the integration lesson as opposed to the differentiation lesson. One student mentioned that it was difficult to engage fully with the activity because he "needed more explanation to figure out the questions." Interestingly, however, there were students that struggled during the differentiation lesson that still reported high levels of lesson engagement. This suggests that students often still engage with LES lessons even if the concepts are challenging. Additionally, student perceptions of engagement can increase, provided the structure of the lesson effectively enables students to grapple with challenging concepts.

Student perceptions of learning are more varied than perceptions of engagement. The majority of students felt like the visual aspects of both lessons helped understand central concepts, and that actively working to understand the material was more effective than "just memorizing how to do something." Students also appreciated that both lessons attributed meaning to integration and differentiation. As one student mentioned, the lessons "show us what we're doing [rather than a problem] telling us do to this equation to figure out an answer." Another student offered similar input, noting that she "likes seeing connections like the ones we made in class." Like the engagement feedback, more students felt that the integration lesson was better for learning as opposed to the differentiation lesson. Specifically, students felt the differentiation lesson was more confusing and therefore not as helpful for learning. A Shepherd student reported that "it was hard to learn since I couldn't understand it," indicating a need for clarity within the lesson. Since students are working independently for the majority of class, the expectations for an LES lesson should be communicated and understandable.

Finally, some students felt that the lessons were not helpful for learning because they differed from traditional instruction. One student commented that she "didn't see the point [of the lessons] and was frustrated. Eventually we'll get an equation and then it's all algebra." Such a response is intriguing because it appears that this student is so accustomed to traditional instruction that any variation in teaching methodology is not important. It also highlights what traditional instruction emphasizes, and how little student engagement and active thinking is present in class on a daily basis.

Overall, the data for this research question supports the hypothesis that the majority of students would positively perceive LES instruction, but engagement feedback would be more positive than learning feedback. It is encouraging that students were still engaged with the activities even if the concepts were not completely clear. Working to understand these concepts in the collaborative LES environment increased positive feedback on learning, indicating the need for precise language choices for activities. While the majority of students also found LES instruction to be effective for learning, there was a slight disconnect between it and student perceptions of traditional instruction. Most students reported that they would enjoy and prefer to see LES instruction more in classes, but some were familiar with traditional teaching and instead preferred a more procedural approach to learning. It is suspected that, if students were exposed to LES instruction consistently from the beginning of the school year, such opinions would dissolve.

Research Question 3

The final research question again asks about student engagement and learning, but this time from the perceptions of the classroom teachers. Both teachers reaffirmed their commitment to traditional instruction when discussing LES. Given that this is the daily structure of their classes, such a result was not incredibly surprising. However, despite reporting that LES instruction was effective in promoting students' critical thinking, understanding of integrals, and understanding of derivatives, both teachers continued to preface their responses with a concern about "the time it takes" and the difficulties in "getting through the entire curriculum." It is telling that, even when presented with "absolute" evidence that LES instruction helped students engage with and learn calculus concepts, teachers who are accustomed to traditional instruction will remain consistent in their methodologies. Rather, both teachers agreed that LES activities would better serve as a "compliment to traditional teaching." In keeping with this trend, it is worth noting that Troy said he was going to "steal from

Mr. Mattis. I'm going to do this" with regard to the integration lesson despite his commitment to traditional instruction.

Additionally, there is an interesting disconnect between student perceptions of engagement and learning and teacher perceptions of engagement and learning. In their feedback, Ray reported that "students may get frustrated with constantly 'discovering' a concept" while Troy believed that "a lot of students by the third or fourth time of doing an LES lesson would get tired of it." These responses are in stark contrast to the student feedback that was gathered. In fact, students reported that they would rather see LES instruction more in mathematics and other classes because "there aren't a lot of classes that make you do hands-on things and get up with people." Moreover, several students commented on how it was refreshing to "come up with mostly everything on our own." From these results, it appears that both classroom teachers thought that students would grow tired of LES instruction when in fact students were asking for more of this type of teaching. Such a disconnect reaffirms the idea that student input is valuable within the classroom, and that listening to students and their perceptions of engagement and learning can provide key implications for future instruction. In this case, it appears that more communication is necessary between teachers and students, and that traditional teachers should be open to a sustainable, student-based method of teaching such as LES. Professional development for LES may also be useful in helping traditional teachers transition to the new method of instruction.

A Return to the Literature

In this section, comparisons between the obtained data and expected findings of the literature are discussed. Specifically, this includes an analysis of student responses to LES instruction as compared to their traditional classroom instruction and teacher perceptions, how the structure of each LES lesson compared to the findings of PBL, and how student perceptions of differentiation and integration compare with the procedural themes found in the literature.

LES Instruction vs. Traditional Instruction

Prior to teaching at either school, I had the opportunity to observe both classroom teachers and typical class routines. Both instructors taught in a style that mirrored the findings of the literature, as they would directly explain a concept before assigning a series of practice problems for students to complete. Students would finish these textbook problems for homework and discuss their answers in the following class period, at which point the cycle generally repeated. Such a trend is consistent with the literature on current calculus teaching methods, as lecture and textbook assignments tend to dominate instruction.

During both of my lessons, there were a small number of instances where the classroom teachers actually interacted with students. Student responses drastically changed when they were discussing with the classroom teachers compared to when they discussed with me or other students. For example, notice the following conversation between the Davenport teacher, Troy, and two students during the differentiation lesson.

T: Did you understand it?
S7: I was getting there, yeah.
T: So the cube now think of it the same way. So to do the cube, because when you do the cube, do you agree this is x^3?
S7/S8: Yeah.
T: So to make it, wouldn't we have 3 lengths?
S7/S8: Yeah.
T: Well actually it'd be going that way. So if we increase it by 1 like you wanted to do, well we'd have to put that block in to find the new height. That little block is your difference. And as that gets smaller and smaller that goes away and so we'll still have 3 blocks. So that's why the derivative of $x^3$is $3x$.

Observe that, even though S7 notes that she is actively working to understand a component of the lesson, Troy immediately begins explaining the concept to her. In turn, both S7 and S8 give very submissive responses, and any indication of active thinking vanishes. Such a pattern is indicative of common classroom tendencies, and unfortunately confirm the existing trends of the literature. It is also noteworthy that Troy incorrectly interprets the problem, claiming that the derivative of x^3 is $3x$. He later returned to this group to correct the mistake.

Comparing LES and PBL

Given the literature gap surrounding LES instruction, research on PBL was utilized as a comparable source of information. Students in PBL environments showed an increase in critical thinking, knowledge transfer, problem-solving, and justification skills while retaining the procedural skills found in traditional classrooms. While this project includes only as a sample of four LES lessons, the data supports this finding. Table 8 and Table 9 indicate that, of the 1,412 student comments and questions throughout all four lessons, 48% are direct instances of Active Thinking, Mathematical Inquiry, or Building Connections. It is noteworthy that the 1,412 comments also contains 503 general student responses, which in some instances occur during periods of critical thinking, teamwork, and justification as well. Additionally, it is worth noting that a significant number of the General Response codes were used for students' measurements in the integration lessons. This lesson component was necessary for students' approximations and still discussion based, but many of the comments were low cognitive demand measurements of each shape. Overall, this small sample of LES instruction seems to support the trends originally found in PBL instruction.

Students' Procedural and Conceptual Knowledge

The final component of the literature that became apparent in analysis was the distinction between students' procedural and conceptual knowledge of the derivative and integral. Research findings indicate that students are more adept at procedural skills involved with differentiation and integration and tend to struggle with conceptualized notions of these topics. Although this trend emerged in the data, it was not as drastic as the literature presented. Students certainly reported for instance that the definition of the power and product rules were the actual procedures, but the LES activity quickly revealed the true meaning of both rules. As a result, students gained a greater conceptual understanding of the derivative, lessening the observed gap in the literature.

Students made similar strides in the integration lesson. Although this lesson was taught as part of an introduction to the integral, the Shepherd class had briefly begun studying this concept prior to my teaching. Following the order of a textbook comparable to Textbook 1 discussed previously, Shepherd students began by calculating definite integrals using the fundamental theorem of calculus without ever discussing the meaning of an integral or the rationale behind the theorem. Rather, they memorized a procedure and used it to solve decontextualized problems. In the classroom teacher feedback form, the Shepherd teacher reported that "to this day, the students in the class that Nate completed an integration activity with connect definite integrals with area under a curve much better than the class he did not teach." Again, this result aligns with the literature, as students in traditional calculus environments were more adept at the procedures involved with integration while students in an LES environment were more comfortable with the conceptual meaning of the integral.

Limitations

This project had several limitations given its structure and logistics. The first limitation involved the classes that were taught. Both classes had similar demographics and structure, and therefore it was not possible to discuss how LES instruction appealed to various students. Rather, classes were selected based on the fact that they were not AP courses and therefore not logistically restricted by the AP exam. It is therefore possible that different students, such as those with learning accommodations, socioeconomic barriers, or behavioral tendencies, may perceive LES instruction in a different manner than what was found in this project.

The second limitation is the lack of community between me and both classes. Although I observed both classes several times prior to teaching, discussed both the project and general mathematics with the students, and included community-driven comments in my teaching, it was not possible to build classroom community from the beginning. Both lessons were taught in February and March which are two months that are well into the school year. Consequently, I could not develop the strong relationships with students that I would have liked. Given the collaborative nature of LES teaching, classroom community and student comfort are critical in effective instruction. Since I did not have the ability to establish this, it is likely that my LES instruction was not as rich as it could have been with a more developed classroom community.

The third limitation is related to analysis and is concerned with the technological difficulties encountered in some of the lessons. In the Shepherd integration lesson, the audio devices for each individual group failed, and therefore a significant portion of data was lost. Specifically, it was not possible to fully analyze student conversations for instances of Active Thinking and Mathematical Inquiry within groups. Instead, only conversations that either the classroom camera or my individual microphone recorded could be analyzed. As a result, it is likely that more relevant examples of student discourse occurred in the Shepherd integration lesson.

Implications

In this section, the results of this project that can be used for future mathematical practice and mathematics education research are discussed. These implications can also help overcome the limitations of this particular study.

For Practice

Perhaps the most obvious implication for practice is the creation and testing of two thorough calculus lessons. Both lessons, aside from the actual manipulatives needed for each activity, are complete and available for use. Additionally, these lessons cover two of the most important topics in calculus, and these topics are also commonly difficult for students in traditional learning environments. Teachers can use the results of this study to adjust the presentation of each lesson if desired, and modify the content to best coincide with their classes.

One of the most significant implications for mathematics education is the reported student perceptions of LES instruction. Contrary to the belief of classroom teachers, the majority of students found LES teaching to be engaging and effective for learning integration and differentiation. Students also enjoyed learning in this manner, making the overall learning experience more productive and meaningful. Teachers can use this information as a basis for lesson structure, as students are not only enjoying such a strategy, but deeply learning critical concepts of calculus as well.

Finally, teachers can feel confident migrating from a traditional, teacher-focused classroom to a more dynamic, student-centered classroom. It is noteworthy that students who learned in a Connected Mathematics Project (CMP) classroom with a less experienced teacher who attended more professional development had higher mathematics scores than their non-CMP counterparts (CMP, 2018). This indicates that, with LES-focused professional development, teachers with varied experiences can effectively utilize LES instruction. While the transition may be difficult or unnatural at first, especially for long-standing traditional teachers, this project offers reassuring evidence that students can thrive at the center of learning. Not only does discourse and engagement increase when students are the main generators of information, but conceptual understanding does as well. This study suggests that students are more than capable of discovering and seriously understanding conceptual aspects of calculus. Teachers should be equally confident in their ability to facilitate such an outcome, especially given the positive results of this project.

For Research

Although the results of this study are encouraging for LES instruction, they are accompanied by a series of future research questions that should be addressed. The first is a more longitudinal study of LES instruction in high school calculus. While a series of two LES lessons taught in two different classrooms produced promising results, a more sustained study of LES teaching would be useful. This would especially address the concerns raised by classroom teachers that LES would not be effective as the basis for a curriculum. Projects that assess the validity of these curricular claims are encouraged.

The second research implication involves LES instruction in other disciplines. For mathematics education, LES could be tested in other high school courses such as geometry, trigonometry, and statistics. For educational research in general, it would be interesting to modify the structure of LES to best fit classes in the sciences, social studies, and arts. Although ambitious, it would be incredibly telling to compare the performance of a school district taught entirely using LES to a traditional school district. Regardless of the scale, research of LES in other classes is necessary to fully assess its effectiveness as an instructional method.

Lastly, additional research is needed on the perceptions of educators who teach using the LES method. This study is focused on my perceptions having taught with this method, but the perceptions of other LES instructors regarding the method's effectiveness should be gathered and analyzed. Such results could uncover the common aspects of each methodology to gain a greater understanding of why certain LES components are effective and others are not.

Conclusion

The overall goal of this research study was to determine the effectiveness of the Launch-Explore-Summarize method of instruction in calculus. Specifically, it explored the various types of engagement that students display throughout an LES lesson on integration and another on differentiation. Additionally, it focused on both student and teacher perceptions of engagement and content learning. Overall, the results of the study are encouraging, and indicate that LES instruction is effective in promoting high levels of student engagement as well as cognitively rich student discourse, thought, and inquiry about the essential components of integration and differentiation. More research is needed to fully understand the intricacies of LES, but this project provides a solid foundation. In addition to learning about an engaging and effective method of teaching calculus, readers hopefully recognize and appreciate the essential bridge between students and teachers, as well as rediscover the intrinsic love of learning that we all possess.

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APPENDICIES

Appendix A: The Common Core Standards for Mathematical Practice

The Common Core Standards for Mathematical Practice

SMP Number	SMP Description
1	Make sense of problems and persevere in solving them
2	Reason abstractly and quantitatively
3	Construct viable arguments and critique the reasoning of others
4	Model with mathematics
5	Use appropriate tools strategically
6	Attend to precision
7	Look for and make use of structure
8	Look for and express regularity in repeated reasoning

Appendix B: Student Assent Form

Student Informed Consent Agreement

Please read the following agreement carefully before choosing to sign.

Hello!
My name is Nate Mattis, and I am currently an undergraduate student at Bucknell University. I am pursuing a degree in mathematics, and also earning my secondary teaching certification. I am currently working on an honors thesis project studying how students react to a specific teaching method. Over the summer, I participated in a teaching program at Brown University where we used the Launch-Explore-Summarize (LES) method of instruction.

I will be teaching 2 LES based lessons in your calculus class. **Please note that these lessons will be videotaped and audio recorded for later analysis in my research.** In no way will this data affect your grades or performance in the class. Your work and contributions will only be used if you choose to participate in the research. In order to protect confidentiality, I will assign you a pseudonym, as I am ultimately concerned with the lesson itself and not your personal information. Even if you choose not to be a part of the research study, you may still participate the lessons that I teach, but I will not analyze your data.

In addition, I will ask a subset of the class to meet with me following the lessons for time to debrief the experience and share your thoughts on the teaching method. No one outside of the discussion group other than the researchers will know what you share.

I am extremely excited to begin working with your class, and am hopeful that my research will help math education overall! If you have any questions at any time, please do not hesitate to contact me at njm013@bucknell.edu . You may also contact Matthew Slater, the chair of the Bucknell Institutional Review Board, at mhs016@bucknell.edu. Thank you!

When I sign my name, this means that I agree to participate in the study and that all of my questions have been answered. I have also been given a copy of this form.

Signature________________________ **Date**________________

Appendix C: Parent Consent Form

Parent/Guardian Informed Consent Agreement

Please read the following agreement carefully before choosing to sign.

Hello! My name is Nate Mattis, and I am currently an undergraduate student at Bucknell University. I am pursuing a degree in mathematics, and also earning my secondary teaching certification. I am currently working on an honors thesis project studying how students react to a specific teaching method. Over the summer, I participated in a teaching program at Brown University where we used the Launch-Explore-Summarize (LES) method of instruction. This method focuses heavily on discovery-based learning, and allows students to solve math problems in a variety of ways and then share their findings with the class in a way that promotes communal growth and learning. I became quite interested in seeing how this method of instruction would work for students in high school calculus classes.

In order to meet this goal, I will be teaching 2 LES based lessons in your child's calculus class. Your child may also be chosen to debrief with me following the completion of the lesson. Your child's education is my primary focus, and I will strive to ensure that the lessons are both effective in learning calculus, and engaging for the class as a whole. **Please note that these lessons will be videotaped and audio recorded for later analysis in my research.** In no way will this data affect your child's grades or performance in the class. Your child's work and contributions will only be used if you choose to participate in the research. In order to protect confidentiality, I will assign a pseudonym for your child. I am ultimately concerned with the lesson itself and not your child's personal information. If you choose not to participate in this research, I will not use your child's contributions anywhere in my research, and will destroy any previously collected data if you opt out of the research project. While I cannot remove your child from the actual class, I will remove any ties to the project.

I am extremely excited to begin working with your class, and am hopeful that my research will help math education overall! If you have any questions at any time, please do not hesitate to contact me at njm013@bucknell.edu. You may also contact Matthew Slater, the chair of the Bucknell Institutional Review Board, at mhs016@bucknell.edu. Thank you!

___ I agree for my child to participate in all aspects of the study

___ I agree for my child to participate in the study, but prefer not to have their face in the video

___ I choose for my child to not participate in this study

Parent Signature___________________________ Date_______________

Appendix D: LES Integration Lesson Plan

Lesson Plan: Integration using LES

Grade Level: High school calculus

Teacher(s): Nate Mattis

- **Mathematical Goal(s):** Students will develop strategies for approximating area under a curve, discuss which strategies are the most effective and why, and begin developing a formal understanding of these processes

- **Related Standards of Mathematical Practice from CCSSM:**
    - Use appropriate tools strategically: Including meter stick and data tables to make accurate and detailed measurements
    - Look for and express regularity in repeated reasoning: Students will compute a variety of measurements using multiple shapes (i.e. rectangular, triangular, circular, etc.) to determine which is most effective. After doing so, students may notice shortcut of using same shape over and over, and therefore could create a formula that models the physical modeling process.
    - Construct viable arguments and critique the reasoning of others: During the discussion of which techniques are most effective, students will need to make conjectures and defend these conjectures. Additionally, students will hear arguments from other groups, and have the opportunity to test these arguments.

- **Related Advanced Placement Calculus AB Standards**
    - L.O 3.2A (a): Interpret the definite integral as the limit of a Riemann sum
        - E.K 3.2A1: A Riemann sum, which requires a partition of an interval I, is the sum of products, each of which is the value of the function at a point in the subinterval multiplied by the length of that subinterval of the partition.
    - L.O 3.2B: Approximate a definite integral
        - E.K 3.2B1: Definite integrals can be approximated for functions that are represented graphically, numerically, algebraically, and verbally.
        - E.K 3.2B2: Definite integrals can be approximated using a left Riemann sum, a right Riemann sum, a midpoint Riemann sum, or a trapezoidal sum.
    - L.O 3.4D: Apply definite integrals to problems involving area.

- **Assessment**
  - Informal during lesson: As students test their theories and perform their measurements, I will be able to observe their procedures and quickly look at data tables. In doing so, I can assess early conjectures and the logic behind these conjectures, and observe if students are making adequate progress and contributions to our overall goals.
  - Formal on students' progress on objectives: This will be done via an exit ticket, which addresses specific content standards, and our full class discussion in the summarize phase of the lesson. The exit ticket will ask which method of measurement was most effective and why, and the discussion will be largely geared toward this same idea and the goals of the lesson.

- **Prior Knowledge**
  - Build on previous knowledge: Students need to utilize their knowledge of basic geometric areas as well as basic measuring skills
  - Needed definitions, concepts, or ideas: We will need to first understand what is meant by "area of a function" as opposed to "area of a geometric shape," and then how these concepts can be related to solve our ultimate problem.

- **Materials:** String for curves, tape for axis, meter sticks, data tables/sequence of questions, "do-now" review of geometric areas, exit ticket, objects to measure with (perhaps cardboard shapes or classroom items)

- **Anticipating what Students will do before the Lesson**
  - All ways that the task can be solved:
    - Technically, any of the shapes can be used to solve the problems, but some will be more effective than others of course. Students may orient the shapes in multiple ways (i.e. a rectangle could be oriented so that its longer side is parallel to x-axis vs y-axis). If students reach the stage of developing formulas, they could be represented differently. For instance, some groups may factor out a common measurement while others include it in every term of the calculation.
  - Which methods will students use?
    - I think students will orient their shapes or objects in such a way that maximizes the area covered by the object. This means that, if a triangle is being used for measurement, students may rotate the triangle to "fill in gaps" as opposed to repeating an orientation over and over. With regard to formula construction, my guess is that students will initially repeat factorable terms in their representation, but could simplify their calculation with a bit of prompting.

- Possible Misconceptions
  - That there is only one correct answer or technique to solving this problem, and how we can create "an infinite number of shapes" to approximate with.
- Possible Errors
  - My biggest concern is measurement errors or incorrect recordings of data

- **Phase One: The Launch**
  - Prior to/as students enter the classroom, there will be several shapes (circle, triangle, rectangle, trapezoid, and parallelogram) around the room. Students will work in small groups to compute the area of each using measurements. Meter sticks will be made available to complete the measurements.
  - A short class discussion will ensue to discuss comfort levels with this process. We will discuss potential reasons for the importance of this activity in calculus. The discussion will conclude with a formal class conjecture about what the "area of a function" means. We would write down our class goals for the lesson on the board to refer back to throughout the lesson.
  - Students will be made aware of the various curves placed around the classroom as well as resources available for measurement (i.e. shapes from do-now). Data tables will be distributed and groups will be assigned (note: I have "random" ways of doing this if I have the opportunity to get to know students prior to this lesson, and therefore can assign specific groupings), before each group is left to explore their curve.
  - Confirm directions by asking for groups to quickly summarize the directions and goals of the lesson prior to the exploration

- **Phase Two: The Explore**
  - As students are working independently, in pairs, or small groups, what questions will you ask to focus their thinking?
    - What is your rationale for orienting your objects in this way?
    - Are there any ways you could simplify your procedure to increase efficiency? (given answer) Why do you think so?
    - Are there any patterns you can find in your data table? How do these patterns relate to our goals?
    - (Later in lesson) Can you generate some conjectures using your data that reflect the goals of our lesson?

  - What will you see or hear that lets you know how students are thinking about mathematical ideas?
    - Early discussion about which procedures work more effectively

than others, possible orientations for the shapes that increase accuracy of approximations, shortcuts to measurements.

- What questions will you ask to assess students' understanding of key mathematical ideas, problem-solving strategies, or representations?
  - How do you know your measurements are effective and accurate?
  - What patterns can you find in your data that could be useful in understanding our goals?
  - Can you discuss/test any strategies that could make your measurements more accurate? Why do you think these strategies will do this?

- What questions will you ask to advance students' understanding of the mathematical ideas?
  - Are there any topics we've talked about this year that could be used to formalize the conjectures you're making? (i.e. limits to get to integrals)
  - How can we generalize these processes to any curve?
  - How do your findings relate to our goals?

- What questions will you ask to encourage students to share their thinking with others or to assess their understanding of their peers' ideas?
  - How do their conjectures relate to yours? Are they the same ideas? Why or why not?
  - Can we test their conjectures?
  - Are both answers valid? Why or why not

- **Phase Three: The Summarize**
  - Which solution paths do you want to have shared during the class discussion in order to accomplish the goals of the lesson?
    - Depending on progress of the groups, I would like to have at least one demonstration of rectangular approximation, and how it compares with other group approximations using other shapes. Additionally, perhaps later in the discussion, I would have groups that thought about/tested how to improve approximation accuracy share their findings.
  - What will you see or hear that lets you know that students in the class understand the mathematical ideas being shared?
    - After a conjecture or answer is posed, I could ask each group to take a minute to discuss it with their group members (while I listened and circulated), and then report back on whether or not they agree with the conjecture and why.
  - What questions will you ask and statements will you make so that students:
    - Make sense of the mathematical ideas being shared?

        - Make connections between their solution strategy and the one shared?
        - Look for patterns and form generalizations?
            - How can we finalize our conjectures to reach a formal and complete approximation?
            - Note that each of your approximation methods are valid! Some are just more effective and accurate than others. For instance, can I approximate the area with just one shape? How does this affect my approximation? Is it still valid?
            - Summarize and discuss the proposed conjecture within your individual groups.

- **Differentiation**
    - For high achieving students, I could challenge them to develop a formula that represents their measurements. Additionally, I could encourage them to use their previous calculus knowledge to transition their formula from a specific case to a generalized and formal representation.
    - For students that struggle, I could have them focus solely on the measurement process and which shapes work better than others. Instead of prompting a more formal discussion (at least at first), let's just understand what exactly is going on when we measure and whether or not circles are easier to work with than triangles (just an example)
    - There are no accommodations for special needs students as per the classroom teacher.

- **Justification for High Cognitive Demand**
    - Students will be required to incorporate a variety of techniques and tools to reach a common goal. They will be utilizing several measurements and have to make inference on a data set, and then transition this concrete data to a more generalized and abstract representation. Finally, students will need to make conjectures about things they possibly cannot test (i.e. infinite rectangles).

- **Justification for Mathematical Discourse**
    - Much of the lesson will be discussion based, whether this be in small groups or with the entire class. The conjectures that students make will necessarily be grounded in mathematical discourse terminology. Additionally, it will be emphasized that these are conjectures, and therefore it is perfectly okay for them to be incorrect. I will make the point that learning a conjecture is wrong is just as important as learning that it is correct. As a result, each conjecture will be regarded solely as a conjecture, and testing it can help make our decision. Doing so will help increase the discourse community and progress overall.

Finding the "Area of a Function"

As you explore your functions and measurements, record your findings the data table below. Have fun!

Shape	**Area of Shape**	**Area of Function**	**Reactions to using Shape**	**Suggestions for Improvement**
Circle				
Rectangle				
Trapezoid				
Triangle				
Parallelogram				

Appendix E: LES Differentiation Lesson Plan

Lesson Plan: Differentiation Using LES

Grade Level: High School Calculus

Teacher(s): Nate Mattis

- **Mathematical Goal(s):** Students will geometrically understand the intuition and logic behind seemingly procedural derivative rules, including the power rule and product rule, and compare these understandings with their prior knowledge of the derivative techniques.

- **Related Standards of Mathematical Practice from CCSSM:**
    - Use appropriate tools strategically: Including base-10 blocks to visually depict x^2 and x^3, as well as tangible changes in area.
    - Reason abstractly and quantitatively: Working with actual functions will allow students to make conjectures about the derivatives of these functions, which can then be applied to general power rule and product rule situations.
    - Look for and make use of structure: Students can notice the underlying structure of how multiplication can be represented as area and how the derivative corresponds to a slight change in area when the side of the structure changes slightly.

- **Related Advanced Placement Calculus AB Standards**
    - LO 2.1C: Calculate Derivatives
        - EK 2.1C3: Sums, differences, products, and quotients can be differentiated using derivative rules.
    - LO 2.3A: Interpret the meaning of the derivative within a problem
    - LO 2.3D: Solve problems involving rates of change in applied contexts
        - EK 2.3D1: The derivative can be used to express information about rates of change in applied contexts.
- **Assessment**
    - Informal during lesson: I will be able to check the progress each pair makes on the guided sheet, and discuss the questions on the sheet with the students. Additionally, I can ask for demonstrations involving the manipulatives coupled with explanation.
    - Formal on students' progress on objectives: This will be done via an exit ticket, which addresses specific content standards, and our full class discussion in the summarize phase of the lesson. The exit ticket will ask which method of measurement was most effective and why, and the discussion will be largely geared toward this same idea.

- **Prior Knowledge**
    - Build on previous knowledge: Students need to utilize their knowledge of geometric areas as well as their already learned knowledge of the power and product rule
    - Needed definitions, concepts, or ideas: We'll need to understand how to visually represent a function in other ways aside from a graph. Additionally, we may need to discuss how to abstractly represent area using functions (i.e. what is $x^2\sin(x)$ as a picture?)

- **Materials:** Guided sheet, base-10 blocks, scratch paper if necessary

- **Anticipating what Students will do before the Lesson**
    - All ways that the task can be solved:
        - Some students may attempt to draw the functions in terms of area while others use the manipulatives to represent. A combination of these strategies is also an option, as some functions are easier to draw than others.
    - Which methods will students use?
        - I think students will use a mix of manipulatives and drawing, but lean on the manipulatives. This technique allows students to physically see which pieces were added and how the area changes as a result of this addition
    - Possible Misconceptions
        - Since the base-10 blocks are not variable, students may perceive the change in area as some constant amount. It will be important to make this clear in the beginning: that the blocks are here as an aid and not necessarily abstract.
    - Possible Errors
        - Students may incorrectly add area or volume to the original function, or interpret the variables in the wrong context.

- **Phase One: The Launch**
    - When students enter the room, a do-now will be presented that asks them to write definitions of the power and product rule, and create an example of each rule.
    - Students will then share their definitions and examples with a neighbor, and based on their conversation, formalize a definition of each rule.
    - Each pair will then share their definitions out to the class and we will collectively decide which final definition we want to use for each rule
    - I anticipate these being largely procedural, at which point I will challenge students to interpret each rule in a different, non-

procedural way

- Pairs will be assigned (“randomly” if possible), materials will be made available and explained, and the guided sheets will be distributed
- Confirm directions by asking for groups to quickly summarize the directions and goals of the lesson prior to the exploration

- **Phase Two: The Explore**
  - Questions to focus thinking during student work
    - What is your rationale for defining the functions in this manner?
    - What do each of your variables represent? Can you show me these in your model?
    - Will you explain your thought process so far? Where do you plan to go next?
    - What added pieces contribute the most to the change in volume or area? How is this reflected in the terms that remain in your expression?
  - What will you see or hear that lets you know how students are thinking about mathematical ideas?
    - How the visuals translate to actual algebraic expressions
    - How the discoveries in the exploration compare to their already known concepts of the power and product rule
  - What questions will you ask to assess students’ understanding of key mathematical ideas, problem-solving strategies, or representations?
    - How could this process generalize to any function?
    - Why can we ignore these specific terms?
    - Can we apply this process to any constant value?
  - What questions will you ask to advance students’ understanding of the mathematical ideas?
    - What is the actual interpretation of this function?
    - By how much do each of these functions change when we slightly adjust x?
  - What questions will you ask to encourage students to share their thinking with others or to assess their understanding of their peers’ ideas?
    - How do their conjectures relate to yours? Are they the same ideas? Why or why not?
    - Are your representations the same? What can you learn from each one?
    - Are both answers valid? Why or why not?

- **Phase Three: The Summarize**
    - Which solution paths do you want to have shared during the class discussion in order to accomplish the goals of the lesson?
        - Depending on progress of the groups, I would select and sequence the questions on the sheet to groups that feel comfortable explaining the process and/or have an interesting/valuable insight on a particular problem
    - What will you see or hear that lets you know that students in the class understand the mathematical ideas being shared?
        - After a conjecture or answer is posed, I could ask each group to take a minute to discuss it with their group members (while I listened and circulated), and then report back on whether or not they agree with the conjecture and why.
    - What questions will you ask and statements will you make so that students:
        - Make sense of the mathematical ideas being shared?
        - Make connections between their solution strategy and the one shared?
        - Look for patterns and form generalizations?
            - Note that, while you might not do this when taking a derivative in the context of a problem, it is helpful to understand the logic and rationale behind each rule.
            - When the rules were first presented, what backing did you have for each of them? How does this compare now that you've done some exploration?
    - How will you help students reflect back on what they have learned?
        - I would frequently refer back to our class goals and ask students why their work reflects these goals. Additionally, I could ask them to summarize their process so as to not lose sight of the goal and rationale for using it.
- **Differentiation**
    - For high achieving students, I have included a challenge problem that focused on the same process but with a different function. This is certainly not intuitive, and working through it is a bit tricky but rewarding.
    - For students that struggle, I could have them focus solely on the simpler functions and heavily use the manipulatives. I can also pair each step with an algebraic step to ultimately develop the rationale behind each rule
    - There are no accommodations for special needs students as per the classroom teacher.

- **Justification for High Cognitive Demand**
  - Students will be required to incorporate a variety of techniques and tools to reach a common goal. The rules we are focusing on are largely taught in a procedural manner, and often only used in this manner. Asking students to visually justify a rule they likely never thought of visually will challenge their cognitive demand.
- **Justification for Mathematical Discourse**
  - Much of the lesson will be discussion based, whether this be in small groups or with the entire class. The conjectures that students make will necessarily be grounded in mathematical discourse terminology. Additionally, it will be emphasized that these are conjectures, and therefore it is perfectly okay for them to be incorrect. I will make the point that learning a conjecture is wrong is just as important as learning that it is correct. As a result, each conjecture will be regarded solely as a conjecture, and testing it can help make our decision. Doing so will help increase the discourse community and progress overall.

Name:
Date:

A Geometric Exploration of Derivative Rules

With your partner(s), explore the questions below using the manipulatives. Have fun!

- Let's start with a simple function, $f(x) = x^2$. Represent this function using the manipulatives.

- In the context of the manipulatives, what do x and $f(x)$ represent?

- Now imagine increasing x, just a little bit, by some small dx. How does $f(x)$ change? Model this with the manipulatives!

- Let's call the change in $f(x)$, df. Using your interpretations in question 2, try to describe this change algebraically.

- Finally, try to manipulate your expression to obtain a result in how df changes proportionally to dx. What would happen to your expression if your initial increase in x was super small?

- Now compute the derivative of $f(x) = x^2$, and compare this to your findings in the other questions. Jot down your thoughts!

- Now that you've conquered $f(x) = x^2$, try to do the same process with $f(x) = x^3$. Jot down some more thoughts and findings!

- So that was the power rule! Crazy! Time to ramp it up a notch with the product rule! Using the manipulatives or a diagram, try to visually model the product $f(x) = x^2 \sin(x)$.

- Once again, imagine increasing x by some some small dx. How does $f(x)$ and your model change?

- Let's keep calling this change df like before. Using your diagram, try to describe df algebraically.

- Finally, try to manipulate your expression to obtain a result in how df changes proportionally to dx. What would happen to your expression if your initial increase in x was super small? Again, how does this compare to your findings if you simply took the derivative of $f(x)$?

A Challenge: Consider the function $f(x) = \frac{1}{x}$. Using a similar process as before, try to reason through the derivative of f. How does this compare with the derivative you'd get if you just applied the power rule?

Name:
Date:

A Geometric Exploration of Derivative Rules

With your partner(s), explore the questions below using the blocks. Have fun!

Concept 1: The Power Rule.

- Let's say that one of the sticks has length x. How do we make x^2 using the other blocks?
- You've now made the function $f(x) = x^2$. What's the area of this function?
- Now imagine increasing the length of x, just a little bit, by some small dx. How does the area of $f(x)$ change? Try to model this with the blocks.
- What is the area of each new piece that you added in terms of x's and dx's?
- Let's call the change in area of $f(x)$ df. Using the last question, how do we algebraically describe df?
- Finally, try to algebraically manipulate your expression to obtain a result in how df changes proportionally to dx. What would happen to your expression if your initial increase in x was super small? (Hint: Think about if the $\lim dx \to 0$)
- Now compute the derivative of $f(x) = x^2$, and compare this to your findings in the other questions. In the context of the blocks, what does the derivative of $f(x) = x^2$ represent?

- Now that you've conquered $f(x) = x^2$, try to do a similar process with $f(x) = x^3$. If $f(x) = x^2$ dealt with area, what would $f(x) = x^3$ deal with?

- Again, imagine changing x by some little amount, dx. How would $f(x)$ change? Try to model this with the blocks.

- Let's keep calling this change in $f(x)$ df like before. Can you algebraically describe this change in terms of x's and dx's? (Hint: Think about doing this for $f(x) = x^2$. What did we do before, and how does that process change for x^3?)

- Finally, try to algebraically manipulate your expression to obtain a result in how df changes proportionally to dx. What would happen to your expression if your initial increase in x was super small?

- Now compute the derivative of $f(x) = x^3$ and compare this to your findings in the other questions. In the context of the blocks, what does the derivative of $f(x) = x^3$ represent?

Concept 2: The Product Rule

- So that was the power rule! Crazy! Time to ramp it up a notch with the product rule! Using the blocks or a drawing, try to visually model the product $f(x) = x^2 \sin(x)$.

- What does $f(x)$ represent in the context of your model?

- Now let's increase x by some little amount dx like before. How does your model change?

- Let's keep calling this change in $f(x$ df like before. Using your diagram, try to describe df algebraically.

- Finally, try to manipulate your expression to obtain a result in how df changes proportionally to dx. What would happen to your expression if your initial increase in x was super small? Again, how does this compare to your findings if you simply took the derivative of $f(x)$?

A Challenge: Consider the function $f(x) = \frac{1}{x}$. Using a similar process as before, try to reason through the derivative of f. How does this compare with the derivative you'd get if you just applied the power rule?

Appendix F: Student Exit Ticket

Feedback Form!

Please answer the questions honestly! Thank you!

1. Briefly summarize the lesson. What did you learn about?

2. Was the lesson engaging? If not, what would you have changed to make it engaging?

3. Do you feel this lesson/teaching style helped you learn? Why or why not?

4. Please write any other comments you have here!

Appendix G: Focus Group Protocol

Focus Group Protocol

The purpose of this focus group is to gather information from a student perspective on the LES method of instruction. Specifically, we hope to understand how students engaged with the lesson and their motivation for doing so, as well as how such a method of teaching compares with traditional and familiar instruction that students receive. This protocol will be largely discussion based and focus on the questions below. However, because this is a discussion format and will include 5-6 students, some questions may be asked that are not listed below, or may be asked in a slightly different way.

Questions:

1. What initial thoughts did you have following the lesson? What did you like and dislike, and why?

2. How does this style of teaching compare with other styles of teaching you have seen? Specifically, have you seen this style of teaching in math?

3. Moving forward, would you like to see this style of teaching more in your math classes? Why or why not?

4. How do you think this style of instruction helped with your understanding of the actual mathematical concepts? Why do you think so?

5. What improvements could be made to this lesson and style of teaching that would help you as a student?

6. Do you have any other questions for me regarding the lesson or research moving forward?

Appendix H: Classroom Teacher Reflection Form

Classroom Teacher Reflection

The purpose of this form is to gather reactions and observational information from an experienced teacher following the LES lessons. This information will be helpful in assessing the engagement of students and comprehension of material from an outside source, and noting aspects of the lesson that may not be apparent from a teaching perspective. I greatly appreciate your responses!

Questions:

1. Did you have any prior knowledge of the Launch-Explore-Summarize (LES) method of teaching? If so, what information did you know?

2. What initial thoughts did you have following both LES lessons?

3. How did you observe your students engage with both lessons? Do you think LES is effective in promoting student engagement?

4. Do you think LES was effective in promoting student learning and comprehension? Why or why not?

5. How might you envision the LES teaching strategy in a long-term setting? Specifically, could this method be incorporated routinely or be the basis for a curriculum? Why or why not?

6. What long-term effects (either positive or negative) might arise from a consistent use of LES teaching?

7. Please include any additional questions or comments here. Thank you again for your responses!

Appendix I: The Final Codebook used for Analysis

Code Name	Code Group	Definition	Example from Davenport Differentiation Lesson Transcript
Classroom Teacher Feedback/Explanation	Feedback/Logistics	This code is used when the classroom teacher either gives feedback about the lesson or explains a concept to a student. In the lessons, I did not explicitly give the classroom teacher participation directions. Rather, many of these comments/questions were individually done.	"Well you do because, let's think about the square. Because it's reducing on this side and this side at the same amount, so you're retaining the square. Does that make sense?" -page 21
Future Lesson Think About	Feedback/Logistics	These are either comments or questions made by any person in the classroom that need to be considered for future LES lessons. They often include things I didn't think about in the lesson planning stage, but could/should be included in another implementation of the current LES lesson.	So I said at the beginning, try not to get bogged down on the numbers. These are all length 10 which makes it difficult. But if I call this some arbitrary length, call it x, how do I make x squared?" -page 7
Negative Lesson Feedback	Feedback/Logistics	This code is solely used for student comments about the lesson that are constructive. They may be made at any time in the lesson and to anyone. Contents of such comments often include structural dislikes, learning imbalances, and the like.	"...just give up. Pretend like you didn't come today." -page 21
Positive Lesson Feedback	Feedback/Logistics	This code is solely used for student comments about the lesson that are positive. They may be made at any time in the lesson and to anyone. Contents of such comments often include structural likes, learning balances, and the like.	"I'm excited about these." -page 4
Off-Topic	Feedback/Logistics	This code is used when anyone in the class discusses something unrelated to the lesson. These comments do not affect the lesson or research.	"It's hollow?" -page 4
General Response SCT	Student Comment Types	SCT refers to "student comment type." This code is used when students give a general answer to a question from anyone in the class. This included myself, the classroom teacher, other peers, or the individual student. Such responses tend to be affirmative and basic, and require the lowest level of thought.	"Kinda." -page 2

Code Name	Code Group	Definition	Example from Davenport Differentiation Lesson Transcript
Active Thinking SCT	Student Comment Types	SCT refers to "student comment type." This code is used when students are working to understand a part of the lesson, whether mathematically or not. These are often early stages of understanding or thought processing, but it's clear that the student is thinking at a more than superficial level. The question must be on topic as well; off-topic questions are not coded. This code is meant to be the "middle" between the mathematical inquiry and gathering info student comment types. It is also often a response to the exploring math meanings/relationships questions and comments.	S11: Whaaat...x is, x is the block. S2: No, x is the side of the block. -page 5
Mathematical Inquiry SCT	Student Comment Types	SCT refers to "student comment type." These comments involve higher levels of student thought about specific mathematical concepts or patterns. The thoughts may be incomplete but active levels of thinking are present. This code involves the highest level of thought. Inquiry is achieved when students comment on one or more of the central foci of the lesson.	S5: Yes. And then I knew that these were *xdx* and this was *dx* squared so I basically took that up there but then added these. S6: That's not what I did. I did what you did up there [on the board] but you add another *xdx*. Because it increases here too. -page 17
Building connections SCT	Student Comment Types	SCT refers to "student comment type." This code is used when students connect the foci of the current mathematical lesson to either different mathematical foci or an entirely different subject of study. At least two distinct topics must be present to build connections between	"We're confused." -page 6
Lesson Confusion SCT	Student Comment Types	SCT refers to "student comment type." This code is used when students specifically comment about the lesson itself. Such comments may involve lesson procedures, directions, or next steps, and may be asked to anyone in the class (me, other students, classroom teacher, or the student themselves). This code is not used for quotes involving math content.	"Oh it would've been so much easier just to do it that way." (referring to using the power rule instead of manipulatives) -page 14

Code Name	Code Group	Definition	Example from Davenport Differentiation Lesson Transcript
Gathering Info SQT	Student Question Types	SQT refers to “student question type.” This code is used when students specifically inquire about procedural or factual components of the lesson. Such questions usually invoke a straightforward response, and may be asked to anyone in the class (me, other students, the classroom teacher, or the student themselves). They involve the lowest level of thought.	"What do you mean by the manipulatives? Is that these things?" -page 5
Active Thinking SQT	Student Question Types	SQT refers to “student question type.” This code is used when students are working to understand a part of the lesson, whether mathematically or not, and asking questions about their thinking. These are often early stages of understanding or thought processing, but it’s clear that the student is thinking at a more than superficial level. The question must be on topic as well; off-topic questions are not coded. This code is meant to be the “middle” between mathematical inquiry and gathering info student question types. It is also often a response to exploring math meaning/relationships comments and questions.	So like x is the variable and f(x) is the function right? -page 5
Mathematical Inquiry SQT	Student Question Types	SQT refers to “student question type.” These questions involve higher levels of student thought about specific mathematical concepts or patterns and are actively being asked either within groups, to the entire class, or to me. Thoughts may be incomplete but evidence of active thinking is present. This is the highest level of thought. Inquiry is achieved when students' questions directly reflect one or more of the central foci of the lesson.	"Could you make that 2xdx?" -page 12
Lesson Confusion SQT	Student Question Types	SQT refers to “student question type.” This code is used when students specifically inquire about the lesson itself. Such questions may involve lesson procedures, directions, or next steps, and may be asked to anyone in the class (me, other students, classroom teacher, or the student themselves). This code is not used for quotes involving math content.	"What do you mean by describe it algebraically? Like with an equation?" -page 8

Code Name	Code Group	Definition	Example from Davenport Differentiation Lesson Transcript
General Response PCT	Personal Comment Types	PCT stands for “personal comment type.” This code is used when I respond to a comment or question from another individual in the class. This may be a student, the classroom teacher, a group, or the class as a whole. These responses tend to be affirmative and basic, such as “okay."	"Okay" -several places
Lesson Directions	Personal Comment Types	This code is used when I explicitly give directions to either the class or specific groups of students. These are often structurally based and involve transitions within the lesson. Lesson directions are not related to content.	Okay, and in your group, you can pair up however you’d like, try to come up with a definition for both the power rule and the product rule. Alright and when you’ve got one, give me a hand or a holler or something and I’ll come over and see what you’ve got, and we’ll go from there. -page 2
LES Explanation	Personal Comment Types	This code is used when I explain either components or purposes of the LES lesson structure. This may be used for transparency within the classroom, rationale for a specific lesson component, or the like. These comments extend beyond lesson directions as their goal is to explain the purpose of a lesson component.	Last time I was here we did a discovery lesson on integration. It wasn’t a topic that you were real familiar with at the time but we kind of used our intuition to guide us where we wanted to go. Today we’re going to do another discovery activity but, based on some of the feedback you gave me last time, it’s going to be on a topic that you’re already pretty familiar with and that I’d be willing to bet is probably pretty easy. -page 1
Affective PCT	Personal Comment Types	PCT stands for “personal comment type.” This code is used when I offer encouragement, praise, or motivation to a student or students. Such a comment may be used to assign competence to a student or simply reaffirm his/her work or attitude. These comments are more than one word responses (ex. right, yeah, nice).	Right! Yeah! -page 5
Exploring Math Meanings/Relationships PCT	Personal Comment Types	This is adapted from the 5 practices book. PCT refers to “personal comment type.” This code is used when I structure my comments or responses in such a way to talk about underlying math relationships and meanings, or to link mathematical ideas and representations.	"Okay, which is exactly what you have right there. I literally make a square"

Code Name	Code Group	Definition	Example from Davenport Differentiation Lesson Transcript
Rephrasing and Extending PCT	Personal Comment Types	PCT refers to “personal comment type.” This code is used when I explicitly structure my responses to students to explore specific mathematical concepts. I use what students say or are working on to form these responses, and also use them to extend student thinking to deeper mathematical concepts or reasoning. There is a distinction between comments and questions, and this code is reserved only for comments. Rephrasing and extending is a step above exploring math meanings/relationships. If a quote can be coded with both, only rephrasing and extending will be used.	Okay and even in terms of the actual blocks. You originally had this, and now you have this. So what’s different? What blocks are different? -page 8
Affective PQT	Personal Question Types	This code is used when I ask a student how he/she feels or his/her personal reactions to something. These questions are not inherently mathematical and instead extend to a more personal basis.	*These are often in the focus group transcriptions*
Gathering Info PQT	Personal Question Types	PQT stands for “personal question type.” This is a question type from 5 practices and are used to invoke a quick and straightforward answer from students. They often focus on the structure of the lesson or current thought process, and are usually asked to invoke quick recall or factual information.	Are we kind of stuck? -page 11
Probing Thinking PQT	Personal Question Types	PQT stands for “personal question type.” This question type is from 5 practices and is used when talking with students either individually or in groups. The purpose of such a question is to have the student(s) expand on current thinking. This is done to either aid in my understanding of student thoughts and/or to subtly introduce additional things for the student(s) to think about.	S6: That is not what I did. N: What did you do? -page 17
Exploring Math Meanings/Relationships PQT	Personal Question Types	PQT refers to “personal question type.” This code is used when I explicitly structure my questions to explore specific mathematical concepts when talking with a student or students. I use what students say or are working on to form these questions. I also use them to relate student thinking to deeper mathematical concepts or reasoning.	And you made a new square out like this. Okay, so if I make these a little bit longer by some dx, what is the area of each of those new pieces that you added?

Code Name	Code Group	Definition	Example from Davenport Differentiation Lesson Transcript
Rephrasing and Extending PQT	Personal Question Types	PQT refers to "personal question type." This code is used when I explicitly structure my questions to explore specific mathematical concepts when talking with a student or students. I use what students say or are working on to form these questions. I also use them to extend student thinking to deeper mathematical concepts or reasoning. Rephrasing and extending is a step above exploring math meanings/relationships. If a quote can be coded with both, only rephrasing and extending will be used.	Yeah. So we're working three-dimensionally. I like that you said we've got this way, that's x, x, dx. But these are the same thing. We have x, how thick is it? -page 17
Generating Discussion PQT	Personal Question Types	PQT stands for "personal question type." This is a question type from 5 practices and are used when I want to invoke a fuller group discussion about a concept or previously raised idea. Group discussion may be with the whole class or within individual working groups.	But originally we said what if this change in x was super small? It's a little bigger in your blocks because they're actually able to be seen, but what if I made this really really small? Very very tiny, and I made my square this way. And even tinier and tinier and tinier. What happens to this term? -page 13 (*these questions were posed to the whole class)*

Appendix J: Student Feedback on Lesson Engagement and Learning

Note: This data came entirely from the exit tickets and focus groups

Lesson	Feedback for Engagement	Feedback for Learning
Davenport Differentiation	"I enjoyed using the shapes (I am a hands-on learner)" "I guess but I didn't know what I was doing" "Yes because I like hands-on learning" "Explain how to start the process so we know what direction to go in"	"Yes because I used a physical figure to help me" "If we didn't already know how to take the derivative before doing this" "I think it was a hard concept to learn" "No" "No, already had an understanding" "Yes and no. We knew the easy way but if we didn't we might have understood" "No I was confused" "Yes. Using things that are there helps me visualize what I am doing" "Yes because being able to see pieces instead of numbers drills it into my brain"
Davenport Integration	"Yes. I was able to really see what I was finding the area of and it tied in well to introducing integrals" "Yes, we had to come up with mostly everything on our own" "Yes, I liked being able to do it myself" "Yeah it was cool. It would have been cool to try the other functions to see how they compare yourself" "It was a fun activity that was more entertaining than sitting at a desk staring at a piece of paper"	"I liked being able to use the shapes. We weren't just given a formula and told to plug numbers in" "I learn better with formulas, numbers, and examples. I like to see something be solved. I like actual answers and not approximations." "I was able to teach myself without knowing. Very creative way." "Yes because I don't understand math in the slightest but I kinda get what was going on here" "Kind of. I wasn't really sure what we were doing at first. I didn't understand the reason as to why we were doing it"

Lesson	Feedback for Engagement	Feedback for Learning
Davenport Integration (continued)	“I liked it because we learning something new but not in a boring way” “It was interesting. I haven’t learned like that in a long time”	“Yes because it was hands on and gave a visual of what we were doing” “I liked having the ability to do it and talk through it” “Instead of just memorizing how to do something we found out why we need to do it” “I like seeing connections like the ones we made in class” “It wasn’t a forced application of it”
Shepherd Differentiation	“Yes” (many exit tickets) “No, need it more understandable” “I thought the use of the blocks made it engaging” “I needed more explanation to figure out the questions” “Yes, very”	“Yes but it was confusing with the blocks because I wanted to associate them with a number rather than the concept itself” “Yes, I would not have been able to figure it out alone” “No, I did not understand” “WHY!!” “Yes, it was a challenge to think about what derivatives really meant” “Yes because it allowed me to visually see what was going on” “No, I couldn’t really follow what was going on” “Kinda, it was hard to learn since I couldn’t understand it”

Lesson	Feedback for Engagement	Feedback for Learning
Shepherd Integration	"Definitely. There aren't a lot of classes that make you do hands-on things and get up with people" "Yes it was fun to work with friends" "Yes it required us to work as a team and actively participate" "Yes" (many exit tickets)	"Working through the problems with physical shapes made the estimation easier to understand. "Yeah it helped me visualize what we were doing" "Yes it helped me learn because it was very hands on" "Yes because I learn by looking and being interactive" "Eh I would of figured it out with or without it" "I didn't see the point and was frustrated by it. Eventually we'll get an equation and then it's all algebra" "It didn't help me with the concept but now I understand what was going on because I could see it go" " It like shows us what we're doing like sometimes it just tells like do this equation to figure out an answer but we actually have to solve to see what we're finding out when we solve an equation

Appendix K: Student Feedback on the Lessons and Future Considerations

Note: This data came explicitly from the Positive Lesson Feedback, Negative Lesson Feedback, and Future Lesson Think About codes.

Lesson	**Positive Feedback**	**Negative Feedback**	**Future Lesson Think About**
Davenport Differentiation	“I’m excited about these” “No I know how to do the problem”	“I don’t like these kinds of questions. Answering a question with a question” “Just give up. Pretend like you didn’t come today” “I mean after today we won’t have to remember it”	“We did not have that in the first place” “My brain’s a little fried after that AP stats test yesterday” “I think the problem for me is I already knew how to do it the other way” “Is this what it feels like to not understand something? I don’t like it” “I think I understand it I just don’t know how to word it”
Davenport Integration	“This is interactive. I enjoy this activity” “I just had a mathematical breakthrough” “What?!” (shock at being able to calculate infinite areas) “Shh, listen to him. Just listen!” “I like the activity because then the notes make sense” “I feel like I’m back in geometry. I love this” “Our function. We should be proud”	“I feel like a guinea pig right now. Like doing this whole experiment thing is weird”	“I feel like it has something to do with derivatives” “I like having the notes first” “I’m getting there. Just a little slow today” “Ah I call not speaking”

Lesson	Positive Feedback	Negative Feedback	Future Lesson Think About
Shepherd Differentiation	“2x. What?!” (shock over the change in area matching the derivative) “Whoa” (same context as above) “It’s nice knowing that it actually means something, the math we’re doing.”	“It’s a trick question” “Honestly, give me an equation. I’ll do that math and figure it out. I’m never going to need this in life” “I mean honestly it’s a little confusing” “I don’t know. I don’t really care” “Oh my gosh. Alright, I quit.”	“I’m really trying to take this seriously but I don’t understand” “2xdx plus...I lost it. That’s what I heard someone say”
Shepherd Integration	“I’d like to see this more.” “It like shows us what we’re doing like sometimes it just tells like do this equation to figure out an answer but we actually have to solve to see what we’re finding out when we solve an equation.”		