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Analysis of Five Diagonal Reproducing Kernels

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ANALYSIS OF FIVE DIAGONAL REPRODUCING KERNELS

by

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Contents

Abstract

The focus of this work is on a specific class of reproducing kernel Hilbert spaces. In Adams and McGuire [2], the tridiagonal reproducing kernels were introduced, and in [3], a specific example of a tridiagonal reproducing kernel Hilbert space was investigated. In particular, a careful functional comparison was made between this tridiagonal space and the well-known Hardy space. This tridiagonal example is studied further in this thesis via the determination of the spectrum of the multiplication by z operator. The main results of this thesis generalize this example to the five diagonal case. A general framework is developed for functionally comparing different bandwidth spaces, and this framework is applied to outline the relationship between the generalized five diagonal example and the Hardy space.

Chapter 1

Introduction

The work in this thesis concerns a specific class of reproducing kernel Hilbert spaces. This section begins by providing a definition for a Hilbert space and giving some examples. Next, a reproducing kernel is defined and the relationship between a reproducing kernel and a corresponding Hilbert space is discussed. Some important types of reproducing kernels that foreshadow the main examples studied in this thesis are highlighted.

Before defining a Hilbert space, some necessary terms are defined. Throughout we will assume that all vector spaces are over the field $\mathbb{C}$ of complex numbers. The unit disk $\{z : |z| < 1\}$ will be denoted by $\mathbb{D}$ and its boundary, the unit circle, by $\partial\mathbb{D}$.

Definition 1.1 (Inner Product)**.** Let V be a vector space. An *inner product*, $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$, is a function satisfying the following conditions for all $x, y, z \in V$ and scalars $\alpha \in \mathbb{C}$:

- $\langle x, x \rangle \geq 0$ with $\langle x, x \rangle = 0$ if and only if $x = 0$;
- $\langle x, y \rangle = \overline{\langle y, x \rangle}$;
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$; and
- $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$.

A vector space equipped with an inner product is called an inner product space. An inner product provides additional structure to a vector space. For example, an inner product gives a way to quantify the interaction between two elements, or vectors, of an inner product space. This gives a sense of "angle" between two vectors in the space. It also allows one to assign lengths to vectors of a vector space via an induced norm. In order to discuss the norm induced by an inner product, first recall the definition of a norm.

Definition 1.2 (Norm)**.** Let V be a vector space. A *norm*, $\|\cdot\| : V \to \mathbb{R}^+$, is a function satisfying the following conditions for all $x, y \in V$ and scalars $\alpha \in \mathbb{C}$:

- $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$;
- $\|\alpha x\| = |\alpha|\|x\|$; and
- $\|x + y\| \leq \|x\| + \|y\|$.

An inner product naturally induces a norm on an inner product space via

$$\|x\| = \sqrt{\langle x, x\rangle}.$$

From this point on, we will assume that this is the norm associated to any given inner product space. An inner product space is complete if every Cauchy sequence of elements in the space converges to an element in the space. Recall a sequence $\{x_n\}$ is Cauchy if for each $\varepsilon > 0$, there is an N such that $\|x_n - x_m\| < \varepsilon$ whenever $n, m > N$.

Definition 1.3 (Hilbert Space)**.** A *Hilbert space* is a complete inner product space.

A collection of vectors $\{f_n\}$ in a Hilbert space is orthonormal if

$$\langle f_n, f_m\rangle = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases}$$

It is well known that every Hilbert space has a basis of orthonormal vectors.

Two Hilbert spaces, H_1 and H_2, are isomorphic if there is a one-to-one and onto mapping between the spaces that preserves the inner product. We will denote such an isomorphism by $H_1 \cong H_2$. Often we will look at Hilbert spaces whose vectors are functions defined on a domain D. When two such Hilbert spaces H_1 and H_2 contain the same functions, but are not necessarily isomorphic, we will denote that relationship by $H_1 \stackrel{s}{=} H_2$. Similarly, if all of the functions in H_1 are in H_2, we will write $H_1 \stackrel{s}{\subseteq} H_2$.

Examples of some well known Hilbert spaces, each with an accompanying orthonormal basis, are listed.

Example 1.4. 1. Let $\mathbb{C}^n = \{(x_1, \ldots, x_n) : x_i \in \mathbb{C}\}$ with inner product defined by

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}.$$

Note that $\mathbb{C}^n$ is a finite dimensional Hilbert space with orthonormal basis $\{e_i\}_{i=1}^n$ where e_i is the n-tuple of complex numbers whose entries are 1 in the i^{th} slot and 0 elsewhere.

2. Let

$$L^2(\partial\mathbb{D}) = \left\{ f : \partial\mathbb{D} \to \mathbb{C} : \int_{\partial\mathbb{D}} |f(s)|^2 ds < \infty \right\}$$

with inner product defined by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{\partial\mathbb{D}} f(s)\overline{g(s)}\ ds$$

where ds denotes arc length measure on $\partial\mathbb{D}$. Note that $L^2(\partial\mathbb{D})$ is an infinite dimensional Hilbert space with orthonormal basis $\{f_n\}_{n=-\infty}^{\infty}$ where $f_n(e^{i\theta}) = e^{in\theta}$. Elements of $L^2(\partial\mathbb{D})$ can be thought of as equivalence classes of square integrable functions on $\partial\mathbb{D}$ where $f \equiv g$ if and only if $f - g = 0$ almost everywhere.

3. Let

$$\ell^2(\mathbb{Z}) = \left\{ \{x_i\}_{i\in\mathbb{Z}} : x_i \in \mathbb{C} \text{ and } \sum_{i\in\mathbb{Z}} |x_i|^2 < \infty \right\}$$

with inner product defined by

$$\langle x, y \rangle = \sum_{i\in\mathbb{Z}} x_i \overline{y_i}.$$

Note that $\ell^2(\mathbb{Z})$ is an infinite dimensional Hilbert space with orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$ where e_n is the sequence indexed by $\mathbb{Z}$ whose entries are 1 in the n^{th} slot and 0 elsewhere. Notice that $\ell^2(\mathbb{Z}) \cong L^2(\partial\mathbb{D})$ via the isomorphism $\phi : \ell^2(\mathbb{Z}) \to L^2(\partial\mathbb{D})$ defined by $\phi(\{a_n\}) = \sum_{n\in\mathbb{Z}} a_n e^{in\theta}$ which identifies the orthonormal basis $\{e_n\}$ of $\ell^2(\mathbb{Z})$ with the orthonormal basis $\{f_n\}$ of $L^2(\partial\mathbb{D})$. It is often convenient to use this identification to interchange back and forth between the two spaces.

4. Let
$$\ell^2_+(\mathbb{Z}) = \left\{ \{x_i\}_{i=0}^{\infty} : x_i \in \mathbb{C} \text{ and } \sum_{i=0}^{\infty} |x_i|^2 < \infty \right\}$$
with inner product defined by
$$\langle x, y \rangle = \sum_{i=0}^{\infty} x_i \overline{y_i}.$$
Note that $\ell^2_+(\mathbb{Z})$ is a subspace of $\ell^2(\mathbb{Z})$ with orthonormal basis $\{e_n\}_{n=0}^{\infty}$.

5. Let $H^2(\mathbb{D})$ be the space of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $\mathbb{D}$ such that $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are in $H^2(\mathbb{D})$, then the inner product of f and g is defined by
$$\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}.$$
Note that $H^2(\mathbb{D})$ is an infinite dimensional Hilbert space with orthonormal basis $\{f_n\}_{n=0}^{\infty}$ where $f_n(z) = z^n$. Also, note that $\ell^2_+(\mathbb{Z}) \cong H^2(\mathbb{D})$ via the isomorphism $\phi : \ell^2_+(\mathbb{Z}) \to H^2(\mathbb{D})$ defined by $\phi(\{a_n\}) = \sum_{n=0}^{\infty} a_n z^n$. As with $\ell^2(\mathbb{Z})$ and $L^2(\partial\mathbb{D})$, we will frequently use this identification to interchange between the two spaces $\ell^2_+(\mathbb{Z})$ and $H^2(\mathbb{D})$. This interchange is well known and both spaces are referred to in the literature as the Hardy space.

The focus of this thesis will be on separable Hilbert spaces, which are Hilbert spaces with countable bases. Some elementary and well known facts about separable Hilbert spaces and the operators on such spaces are included below for completeness. These facts will be used later in this section when defining the reproducing kernel.

Proposition 1.5. *If H is a separable Hilbert space with orthonormal basis $\{f_n\}_{n=0}^{\infty}$, then for each $g \in H$,*
$$g = \sum_{n=0}^{\infty} a_n f_n = \sum_{n=0}^{\infty} \langle g, f_n \rangle f_n.$$
Moreover,
$$\|g\|^2 = \sum_{n=0}^{\infty} |a_n|^2 = \sum_{n=0}^{\infty} |\langle g, f_n \rangle|^2.$$

Proof. Since $\{f_n\}$ is a basis, $g = \sum_{n=0}^{\infty} a_n f_n$ for some sequence $\{a_n\}$ of complex numbers. Since $\{f_n\}$ is an orthonormal basis, the proposition follows on noting that

$$\langle g, f_n\rangle = \left\langle \lim_{N\to\infty} \sum_{j=0}^{N} a_j f_j, f_n \right\rangle = \lim_{N\to\infty} \sum_{j=0}^{N} a_j \langle f_j, f_n\rangle = \lim_{N\to\infty} \sum_{j=0}^{N} a_n = a_n.$$

□

Definition 1.6. Let H and K be Hilbert spaces.

1. A *linear operator is a function $L : H \to K$ such that*

   $$L(\alpha f + g) = \alpha L(f) + L(g)$$

   for all $f, g \in H$ and $\alpha \in \mathbb{C}$.

2. *A linear operator L is* bounded if there exists a nonnegative real number M such that $\|Lh\| \leq M\|h\|$ for all $h \in H$. The smallest such constant M is denoted $\|L\|$ and is called the *norm* of L.

3. A linear operator $L : H \to \mathbb{C}$ is called a *linear functional.*

The following proposition shows that boundedness is equivalent to continuity for a linear operator.

Proposition 1.7. *If H and K are Hilbert spaces and $L : H \to K$ is a linear operator, then L is bounded if and only if L is continuous.*

Proof. First assume that L is bounded. Let M be a constant such that $\|Lh\| \leq M\|h\|$ for all $h \in H$. Then $\|Lf - Lg\| = \|L(f - g)\| \leq M\|f - g\|$. Therefore L is (uniformly) continuous.

Now assume that L is continuous. Choose an element $f_0 \in H$. Since L is continuous at f_0, there exists a $\delta > 0$ such that $\|L(f - f_0)\| = \|Lf - Lf_0\| < 1$ whenever $\|f - f_0\| < \delta$. If $f \in H$ is arbitrary and nonzero, then $\|(\frac{\delta}{2}\frac{f}{\|f\|} + f_0) - f_0\| = \frac{\delta}{2} < \delta$. Hence $\|L(\frac{\delta}{2}\frac{f}{\|f\|})\| < 1$, and so $\|Lf\| \leq \frac{2}{\delta}\|f\|$. Notice that $\|Lf\| \leq \frac{2}{\delta}\|f\|$ holds for $f = 0$. Since f was arbitrary, L is bounded by the constant $\frac{2}{\delta}$. □

The next result asserts that all bounded linear functionals on a Hilbert space have a particularly nice representation.

Theorem 1.8 (Riesz Representation Theorem)**.** *If $L : H \to \mathbb{C}$ is a bounded linear functional on a Hilbert space H, then there exists a unique $g \in H$ such that $L(f) = \langle f, g \rangle$ for all $f \in H$.*

Proof. If $L = 0$, then $g = 0$ is a vector in H is such that $L(f) = \langle f, g \rangle$ for all $f \in H$. Suppose $L \neq 0$ and let $M = \ker(L)$. Note that since $L \neq 0$, $M \neq H$. So $M^{\perp} \neq \{0\}$ and there is a vector $g_0 \in M^{\perp}$ such that $L(g_0) = 1$. If $f \in H$ and $\alpha = L(f)$, then $L(f - \alpha g_0) = 0$. So $f - \alpha g_0 \in M$. Therefore $0 = \langle f - L(f)g_0, g_0 \rangle = \langle f, g_0 \rangle - \alpha \|g_0\|^2$ since $g_0 \perp M$. Hence $L(f) = \alpha = \langle f, \frac{g_0}{\|g\|^2} \rangle$. So $g = \frac{g_0}{\|g\|^2}$ is such that $L(f) = \langle f, g \rangle$ for all $f \in H$.

It remains to show that g is unique. If $h \neq g$ is a vector in H such that $L(f) = \langle f, h \rangle$ for all $f \in H$, then $\langle f, h - g \rangle = L(f) - L(f) = 0$ for all $f \in H$. Thus $h - g = 0$, and we have established uniqueness. □

The Hilbert spaces studied in this thesis are separable Hilbert spaces whose elements are complex-valued functions on a domain D. In this case, it makes sense to define for each $w \in D$, an *evaluation map* given by $E_w(f) = f(w)$. If E_w is bounded, then the Riesz representation theorem implies the existence of a unique function $k_w \in H$ such that

$$E_w(f) = f(w) = \langle f, k_w \rangle$$

for all $f \in H$. This property is called the reproducing property.

Definition 1.9 (Reproducing Kernel Hilbert Space)**.** A Hilbert space H of functions on D is a *Reproducing Kernel Hilbert space* if E_w is bounded for each $w \in D$ and H is the closed linear span of $\{k_w : w \in D\}$.

The fact that the Riesz representation theorem guarantees the uniqueness of k_w for each $w \in D$ allows for the following definition.

Definition 1.10 (Reproducing Kernel)**.** The *reproducing kernel* of a reproducing kernel Hilbert space H is the unique function $K : D \times D \to \mathbb{C}$ defined by $K(z, w) = k_w(z)$.

Since $k_w \in H$, it is true that $K(z, w) = \langle k_w, k_z \rangle$. Therefore the reproducing kernel satisfies

$$K(z, w) = \langle k_w, k_z \rangle = \overline{\langle k_z, k_w \rangle} = \overline{K(w, z)}.$$

Also, the reproducing kernel satisfies the following property which is called positive definiteness:

$$\sum_{i,j=1}^{n} \overline{\alpha}_i \alpha_j K(w_i, w_j) > 0$$

for any finite set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq \mathbb{C} \setminus \{0\}$, and $\{w_1, w_2, \ldots w_n\} \subseteq D$. To see this, notice

$$\sum_{i,j=1}^{n} \overline{\alpha}_i \alpha_j K(w_i, w_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{\alpha}_i \alpha_j \langle k_{w_j}, k_{w_i} \rangle$$

$$= \left\langle \sum_{j=1}^{n} \alpha_j k_{w_j}, \sum_{i=1}^{n} \alpha_i k_{w_i} \right\rangle = \left\| \sum_{i=1}^{n} \alpha_i k_{w_i} \right\|^2 > 0.$$

Since $K(z, w) = \overline{K(w, z)}$, it follows that if $K(z, w)$ is analytic in z, then K is coanalytic in w. It is also well known that

$$\left\{ \sum_{j=1}^{n} \alpha_i k_{w_i} : \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq \mathbb{C} \text{ and } \{w_1, w_2, \ldots w_n\} \subseteq D \right\}$$

is dense in H.

Conversely, if $K : D \times D \to \mathbb{C}$ is positive definite, then there is a unique Hilbert space $H(K)$ such that K is the reproducing kernel for $H(K)$. Namely, $H(K)$ is the closed linear span of K. Notice that the existence of the reproducing kernel guarantees that point evaluation is a bounded linear functional for each $w \in D$. Therefore, $H(K)$ is a reproducing kernel Hilbert space. We could equivalently define a reproducing kernel Hilbert space to be a Hilbert space that contains a reproducing kernel.

The following proposition appeals to Proposition 1.5 in order to obtain a formula for $K(z, w)$.

Proposition 1.11. *If $\{f_n\}_{n=0}^{\infty}$ is an orthonormal basis for a reproducing kernel Hilbert space H, then $K(z, w) = \sum_{n=0}^{\infty} \overline{f_n(w)} f_n(z)$.*

Proof. Let $f \in H$ and $w \in D$. Since $f = \sum_{n=0}^{\infty} \langle f, f_n \rangle f_n$, we have

$$f(z) = \langle f, k_z \rangle = \left\langle \sum_{n=0}^{\infty} \langle f, f_n \rangle f_n, k_z \right\rangle = \sum_{n=0}^{\infty} \langle f, f_n \rangle \langle f_n, k_z \rangle = \sum_{n=0}^{\infty} \langle f, f_n \rangle f_n(z).$$

This holds true for all $f \in H$, in particular it holds true for k_w. Thus

$$K(z,w) = k_w(z) = \sum_{n=0}^{\infty} \langle k_w, f_n \rangle f_n(z) = \sum_{n=0}^{\infty} \overline{f_n(w)} f_n(z).$$

□

Example 1.12. Consider the Hardy space $H^2(\mathbb{D})$. We will show that $H^2(\mathbb{D})$ is a reproducing kernel Hilbert space with kernel $K(z,w) = \frac{1}{1-\overline{w}z}$. Recall that if $f \in H^2(\mathbb{D})$, then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ where $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. Also note that if $w \in \mathbb{D}$, then $\{w^n\}_{n=0}^{\infty} \in \ell_+^2(\mathbb{Z})$ and

$$\|w^n\|_2^2 = \sum_{n=0}^{\infty} |w|^{2n} = \frac{1}{1-|w|^2}.$$

Hence

$$|E_w(f)| = |f(w)| = \left| \sum_{n=0}^{\infty} a_n w^n \right| = \left| \langle \{a_n\}, \{\overline{w}^n\} \rangle_{\ell_+^2} \right|.$$

By the Cauchy-Schwarz inequality,

$$|E_w(f)| \le \|\{a_n\}\|_2 \|\overline{w}^n\|_2 = \|f\|_2 \sqrt{\frac{1}{1-|w|^2}}.$$

Therefore, E_w is a bounded linear functional on $H^2(\mathbb{D})$.

To see that $H^2(\mathbb{D})$ is the closed linear span of $\{k_w : w \in \mathbb{D}\}$, note that if $f \in H^2(\mathbb{D})$ is orthogonal to k_w for all w, then $f(w) = \langle f, k_w \rangle = 0$ for all $w \in \mathbb{D}$. Hence $f = 0$. This shows that $H^2(\mathbb{D})$ is a reproducing kernel Hilbert space. By Proposition 1.11, the reproducing kernel for $H^2(\mathbb{D})$ is given by

$$K(z,w) = \sum_{n=0}^{\infty} \overline{w}^n z^n = \sum_{n=0}^{\infty} (\overline{w}z)^n = \frac{1}{1-\overline{w}z}.$$

We now focus on spaces $H(K)$ whose elements are analytic functions on a connected domain $D \subseteq \mathbb{C}$. In this case, $K(z,w) = k_w(z)$ is analytic in z on D and coanalytic in w. Since $K : D \times D \to \mathbb{C}$ is analytic in the first variable z and coanalytic in the second variable w, K has a convergent Taylor series expansion, $K(z,w) = \sum_{i,j=0}^{\infty} a_{ij}(z-z_0)^i(\overline{w}-\overline{w}_0)^j$, about each point $(z_0, w_0) \in D \times D$. By a translation and a rescaling, we may assume that $(z_0, w_0) = (0,0)$ and consider only kernels of the form $K(z,w) = \sum_{i,j=0}^{\infty} a_{ij} z^i \overline{w}^j$ where D contains the open unit disk $\mathbb{D}$.

The kernel K can be written more compactly by writing $K(z,w) = \overline{\mathbf{z}}^* A \overline{\mathbf{w}}$ where $\mathbf{z}$ denotes the column vector whose transpose is $(1, z, z^2, \ldots)$ and $A = [a_{i,j}]_{i,j=0}^{\infty}$. Recall, a complex $n \times n$ matrix M is positive semi-definite (positive definite) if $\langle Mf, f\rangle \geq 0$ for all $f \in \mathbb{C}^n$ (if $\langle Mf, f\rangle > 0$ for all nonzero $f \in \mathbb{C}^n$). Similarly, the infinite matrix $A = [a_{i,j}]_{i,j=0}^{\infty}$ is positive semi-definite if $\langle Af, f\rangle \geq 0$ for each nonzero $f \in \ell^2_+$. If $\langle Af, f\rangle$ is strictly positive, then A is positive definite. Since

$$\sum_{i,j=0}^{n} \alpha_i \overline{\alpha}_j K(w_j, w_i) = \left\langle A\left(\sum_{i=0}^{n} \alpha_i \overline{\mathbf{w}}_i\right), \left(\sum_{j=0}^{n} \alpha_j \overline{\mathbf{w}}_j\right)\right\rangle$$

and $\{\overline{\mathbf{w}} : w \in \mathbb{D}\}$ has dense span in ℓ^2_+, it is clear that K is positive semi-definite as a function on $\mathbb{D} \times \mathbb{D}$ if and only if A is a positive semi-definite matrix on ℓ^2_+. Therefore K is positive definite if and only if A is positive definite, which is true if and only if $\ker(A) = \{0\}$. Henceforth, A will be assumed to have trivial kernel and $H(A)$ will denote the space $H(K)$ where $K(z,w) = \overline{\mathbf{z}}^* A \overline{\mathbf{w}}$.

Recall that A is positive definite if and only if $A = BB^*$ for some matrix B. Also recall that if B is a bounded operator, then the range space of B is the Hilbert space $R(B) = \{B\mathbf{x} : \mathbf{x} \in \ell^2_+\}$ with inner product $\langle B\mathbf{x}, B\mathbf{y}\rangle_{R(B)} = \langle \mathbf{x}, \mathbf{y}\rangle_{\ell^2_+}$. Note $R(B)$ is a reproducing kernel Hilbert space with

$$K(z,w) = \overline{\mathbf{z}}^*(BB^*)\overline{\mathbf{w}} = \overline{\mathbf{z}}^* A \overline{\mathbf{w}}.$$

Since reproducing kernels give rise to unique reproducing kernel Hilbert spaces,

$$R(B) \cong H(A) = H(K)$$

where $B\mathbf{x}$ is identified with the function $\langle B\mathbf{x}, \overline{\mathbf{z}}\rangle_{\ell^2_+}$. With this identification, Corollary 3.2 of [1] implies that $\{\langle Be_n, \overline{\mathbf{z}}\rangle_{\ell^2_+}\}$ is an orthonormal basis for $H(A)$. Therefore, the columns of B are the coefficient vectors of a basis for the space $H(A)$.

Since A is positive definite, we can use the well known Cholesky algorithm to write $A = LL^*$ where L is a lower triangular matrix. This decomposition makes the identification of a particularly useful orthonormal basis an easy task, as illustrated in the following examples.

Example 1.13. 1. Let $H(K)$ be the reproducing kernel Hilbert space with

$$K(z,w)=\sum_{n=0}^{\infty}a_n(\overline{w}z)^n=\sum_{n=0}^{\infty}(\sqrt{a_n}z^n)(\sqrt{a_n}\overline{w}^n).$$

The associated coefficient matrix A is

$$A=\begin{pmatrix} a_0 & 0 & \cdots & & & & \\ 0 & a_1 & 0 & \cdots & & & \\ \vdots & 0 & a_2 & 0 & \cdots & & \\ & \vdots & 0 & a_3 & 0 & \cdots & \\ & & \vdots & & \ddots & & \end{pmatrix}$$

$$=\begin{pmatrix} \sqrt{a_0} & 0 & \cdots & & & & \\ 0 & \sqrt{a_1} & 0 & \cdots & & & \\ \vdots & 0 & \sqrt{a_2} & 0 & \cdots & & \\ & \vdots & 0 & \sqrt{a_3} & 0 & \cdots & \\ & & \vdots & & \ddots & & \end{pmatrix}\begin{pmatrix} \sqrt{a_0} & 0 & \cdots & & & & \\ 0 & \sqrt{a_1} & 0 & \cdots & & & \\ \vdots & 0 & \sqrt{a_2} & 0 & \cdots & & \\ & \vdots & 0 & \sqrt{a_3} & 0 & \cdots & \\ & & \vdots & & \ddots & & \end{pmatrix}.$$

Notice that $H(K)$ has $\{\sqrt{a_n}z^n\}_{n=0}^{\infty}$ as an orthonormal basis. Spaces $H(K)$ of this form are called diagonal spaces.

2. Let $H(K)$ be the reproducing kernel Hilbert space with

$$K(z,w)=\sum_{n=0}^{\infty}f_n(z)\overline{f_n(w)} \quad \text{where} \quad f_n(z)=(a_{n,0}+\cdots+a_{n,J}z^J)z^n.$$

The associated matrix is

$$A=\begin{pmatrix} a_{0,0} & 0 & \cdots & \\ a_{0,1} & a_{1,0} & 0 & \cdots \\ \vdots & a_{1,1} & \ddots & \\ a_{0,J} & \vdots & \ddots & \\ 0 & a_{1,J} & & \\ \vdots & 0 & \ddots & \\ & \vdots & & \end{pmatrix}\begin{pmatrix} \overline{a_{0,0}} & \overline{a_{0,1}} & \cdots & \overline{a_{0,J}} & 0 & \cdots & \\ 0 & \overline{a_{1,0}} & \overline{a_{1,1}} & \cdots & \overline{a_{1,J}} & 0 & \cdots \\ \vdots & 0 & \ddots & \ddots & & \ddots & \\ & \vdots & & & & & \end{pmatrix}$$

The main results of this thesis concern spaces of this form with $J=2$. These spaces are called five diagonal reproducing kernel Hilbert spaces because the matrix A is nonzero only on the five central diagonals.

In previous work, these spaces were studied and in some cases, $H(K)$ was specifically described. In particular, when $J = 1$, $p > 0$, and $H(K)$ has orthonormal basis

$$f_n(z) = \left(1 - \left(\frac{n+1}{n+2}\right)^p z\right) z^n,$$

$H(K)$ was explicitly described in Admas and McGuire [3]. In section 3, we will consider this space and determine the spectrum of the operator of multiplication by z. Section 2 consists of some well known preliminary results regarding operators on reproducing kernel Hilbert spaces that are needed for sections 3 and 4. In section 4, the work in [2] is naturally extended to the five diagonal reproducing kernel Hilbert space with orthonormal basis $\{f_n\}$ where

$$f_n(z) = \left(1 - \left(\frac{n+1}{n+2}\right) z\right)\left(1 - \left(\frac{n+1}{n+2}\right) e^{i\theta_0} z\right) z^n.$$

Section 5 provides a summative conclusion of the work in this thesis, and section 6 provides a collection of open questions. Section 7 is an appendix of background results in topology and analysis on which the results in section 2 depend.

Chapter 2

Operators on reproducing kernel Hilbert spaces

The results in this section are well known in the literature and most can be found in Conway [5]. The primary focus will be on multiplication operators, which will be defined and motivated shortly.

Theorem 2.1 (Closed Graph Theorem)**.** *If X and Y are Banach spaces and $T : X \to Y$ is a bounded linear transformation such that the graph of T,*

$$\mathcal{G}(T) = \{x \oplus Tx \in X \oplus Y : x \in X\}$$

is closed, then T is continuous.

Proof. Since the direct sum $X \oplus Y$ is a Banach space and $\mathcal{G}(T)$ is closed, $\mathcal{G}(T)$ is a Banach space. Define $P : \mathcal{G}(T) \to X$ by $P(x \oplus Tx) = x$. It is straightforward to check that P is bounded and bijective. By the Inverse Mapping Theorem (see section 7), $P^{-1} : X \to \mathcal{G}(T)$ is continuous. Thus $T : X \to Y$ is the composition of the continuous map $P^{-1} : x \to \mathcal{G}(T)$ and the continuous map of $\mathcal{G}(T) \to Y$ defined by $x \oplus Tx \to Tx$. Therefore T is continuous. □

Definition 2.2 (Multiplier)**.** A function ϕ defined on a set D is a *multiplier* of a reproducing kernel Hilbert space H of functions defined on D if ϕf is in H whenever f is in H. Note that if H contains the constant functions, then any multiplier ϕ of H must be in H.

For $\phi \in H$, define the operator of multiplication by ϕ by M_ϕ where $M_\phi f = \phi f$ for $f \in H$. Note that ϕ is a multiplier of H if $M_\phi(H) \subseteq H$. Multiplication operators are a very well studied and natural class of operators to consider on any function or L^2 space. In linear algebra one shows that a linear transformation T (operator) is diagonalizable if and only if there exists a basis $\{v_1, \dots, v_n\}$ of eigenvectors associated with the eigenvalues $\{\lambda_1, \dots, \lambda_n\}$. In this case T can be written as $T = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_n}$ where E_{λ_i} denotes the projection onto the eigenvector v_i of the eigenvalue λ_i. If the eigenvalues are distinct and m_i denotes the unit point mass measure at λ_i, then the Hilbert space $\mathbb{C}^n$ can be viewed as $L^2(m)$ where $m = m_1 + \cdots + m_n$. In this case the domain of the "functions" in $L^2(m)$ is $D = \{\lambda_1, \dots, \lambda_n\}$, the operator T is identifiable with M_z on $L^2(m)$, and M_ϕ is identifiable with the operator $\phi(T)$. While the situation is much more complicated for general L^2 and function spaces, the motivations to study the multiplication operators are sufficiently strong that identifying the multipliers and understanding M_z is of fundamental importance.

The following result is a consequence of the Closed Graph Theorem and shows that the operator M_ϕ is in fact continuous whenever ϕ is a multiplier and M_ϕ is well defined.

Proposition 2.3. *If $H(K)$ is a reproducing kernel Hilbert space of analytic functions defined on D and $\phi \in H(K)$, then ϕ is a multiplier of $H(K)$ if and only if the multiplication operator M_ϕ is bounded on $H(K)$.*

Proof. Suppose that ϕ is a multiplier of an analytic reproducing kernel Hilbert space $H(K)$ of functions defined on a domain D. The Closed Graph Theorem will be used to show that M_ϕ is bounded. Let $f_n \oplus M_\phi f_n \to f \oplus g$ in $H(K) \oplus H(K)$. Hence $f_n \to f$ and $M_\phi f_n \to g$. The graph of M_ϕ will be shown to be closed by proving $M_\phi f = g$. Note that $M_\phi f$ is well defined since ϕ is assumed to be a multiplier of $H(K)$. Since $f_n \to f$, for each $w \in D$, $< f_n - f, k_w > \to 0$. Thus $f_n(w) \to f(w)$ for all $w \in D$. Since

$$< \phi f_n - \phi f, k_w > = \phi(w) f_n(w) - \phi(w) f(w) = \phi(w)\Big(f_n(w) - f(w)\Big) \to 0,$$

and $\{k_w : w \in G\}$ is a dense subset of $H(K)$, $M_\phi f_n \to M_\phi f$ and $g = M_\phi f$ as desired.

Now if M_ϕ is bounded on $H(K)$, then there is an M such that $\|M_\phi f\| \le M\|f\| < \infty$. Therefore $M_\phi f \in H(K)$ and ϕ is a multiplier. □

Example 2.4. The function $\phi(z) = z$ is easily seen to be a multiplier of $H^2(\mathbb{D})$. Recall $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2(\mathbb{D})$ if and only if $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. Hence $(M_z f)(z) =$

$\sum_{n=0}^{\infty} a_n z^{n+1}$ is in $H^2(\mathbb{D})$ whenever $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is in $H^2(\mathbb{D})$. Moreover, if ϕ is any bounded analytic function with $|\phi(z)| \leq M$, then

$$\int_{\partial\mathbb{D}} |\phi f|^2 ds \leq M^2 \int_{\partial\mathbb{D}} |f|^2 ds \leq M^2 \|f\|_2^2$$

which shows that ϕf is in $H^2(\mathbb{D})$ considered as a subspaces of $L^2(\mathbb{D})$.

Definition 2.5 (Spectrum)**.** The *spectrum of a bounded linear operator T on a Banach space X over $\mathbb{C}$, denoted $\sigma(T)$, is the set $\{\lambda \in \mathbb{C} : T - \lambda I$ is not invertible$\}$ where I is the identity operator.*

Lemma 2.6. *If T is a bounded linear operator on a Banach Space X over $\mathbb{C}$ and $\|T - I\| < 1$, then T is invertible.*

Proof. Let $B = I - T$. Note that $\|B^n\| \leq \|B\|^n \leq r^n < 1$ for some $r < 1$. Therefore $\sum_{n=0}^{\infty} \|B^n\| < \infty$ and $\sum_{n=0}^{\infty} B^n$ converges to a continuous linear operator on X, call it A. If $A_N = \sum_{n=0}^{N} B^n$, then $A_N(I-B) = I-B^{N+1}$. Since $\|B^{N+1}\| \leq r^{N+1}$, $B^{N+1} \to 0$ as $n \to \infty$. Therefore

$$A(I - B) = \lim_{N\to\infty} A_N(I - B) = \lim_{N\to\infty} I - B^{N+1} = I.$$

Therefore, A is a left inverse of $I - B$. A similar argument shows $(I - B)A = I$. So $A = (I - B)^{-1}$. Since $B = I - T$, $T = I - B$, and so T is invertible. □

Proposition 2.7. *The spectrum of a bounded linear operator T on a Banach space X over $\mathbb{C}$ is a closed and bounded subset of $\mathbb{C}$.*

Proof. Let G be the set of invertible bounded linear operators on X and let $L \in G$. Let S be a bounded linear operator on X such that $\|S - L\| < \frac{1}{\|L\|}$. Then

$$\|SL^{-1} - I\| = \|(S - L)L^{-1}\| \leq \|S - L\|\|L^{-1}\| < \frac{1}{\|L^{-1}\|}\|L^{-1}\| = 1.$$

By Lemma 2.6, SL^{-1} is invertible. Therefore, $(SL^{-1})T = S$ is invertible since it is the composition of invertible bounded linear operators. Hence, the set G of invertible bounded linear operators on X is an open set.

Let ϕ be the map from $\mathbb{C}$ to the set of bounded linear operators on X defined by $\phi(\lambda) = \lambda I - T$. Since $\lambda_n \to \lambda$ as $n \to \infty$ implies $\lambda_n I - T \to \lambda I - T$ as $n \to \infty$, ϕ is continuous. Thus, $\phi^{-1}(G) = \mathbb{C} \setminus \sigma(T)$ is open in $\mathbb{C}$. Therefore $\sigma(T)$ is closed.

Let $\lambda \in \mathbb{C}$ with $|\lambda| > \|T\|$. Then $\|(I - \frac{1}{\lambda}T) - I\| = \|\frac{1}{\lambda}T\| < 1$. By Lemma 2.6, $I - \frac{1}{\lambda}T$ is invertible. So $-\lambda(I - \frac{1}{\lambda}T) = T - \lambda I$ is invertible. Thus $\sigma(T)$ is contained in the closed disk of radius $\|T\|$ and $\sigma(T)$ is bounded. □

Definition 2.8 (Spectral Radius)**.** The *spectral radius of a bounded linear operator* T *is* $r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\}$.

The multiplication operator M_z is an operator of particular interest. The spectrum of M_z is determined for a specific example in Theorem 3.3.

When studying a reproducing kernel Hilbert space, $H(K)$, it is useful to know how it compares to other reproducing kernel Hilbert spaces, say $H(K_1)$. Since the space $H(K) = H(A)$ is isomorphic to $R(B)$ where $A = BB^*$ and $H(K_1) = H(A_1)$ is isomorphic to $R(B_1)$ where $A_1 = B_1B_1^*$, we can determine the comparison between $H(K)$ and $H(K_1)$ by determining the comparison between $R(B)$ and $R(B_1)$. The following result of Douglas [6] allows for one to make this determination.

Theorem 2.9 (Douglas' Lemma)**.** *If* T *and* L *are bounded linear operators on a Hilbert space* H*, then* $Range(T) \subseteq Range(L)$ *if and only if there exists a bounded linear operator* C *on* H *such that* $T = LC$*.*

In Section 4, a specific reproducing kernel Hilbert space is studied. Douglas's Lemma is a fundamental tool used to compare that reproducing kernel Hilbert space to $H^2(\mathbb{D})$.

Chapter 3

Tridiagonal Reproducing Kernels

A particularly interesting class of reproducing kernel Hilbert spaces is the class of finite diagonal reproducing kernel Hilbert spaces. It is well known, see Aronszajn [4], that if $\{f_n(z)\}$ is an orthonormal basis for a reproducing kernel Hilbert space of functions on E, then $K(z,w) = \sum_{n=0}^{\infty} f_n(z)\overline{f_n(w)}$ for all z, w in E. Moreover if the largest common domain E' of the functions $\{f_n(z)\}$ is larger than E, then the largest natural domain of $H(K)$ is given by $Dom(K) = \{z \in E' : \sum_{n=0}^{\infty} |f_n(z)|^2 < \infty\}$.

In the very well studied diagonal case where $f_n(z) = a_n z^n$ (see Shields [7]), $Dom(K)$ is always a disk. In Adams and McGuire [2] the domain of functions in the reproducing kernel Hilbert space with orthonormal basis $\{f_n\}_{n=0}^{\infty}$ where $f_n(z) = (a_{n,0} + \cdots + a_{n,J} z^J) z^n$ is shown to be either an open or closed disk about the origin together with at most J points not in the disk. This result not only illustrates a key distinction between the finite bandwidth reproducing kernel Hilbert spaces and the diagonal spaces, but also motivates interest in the role those additional domain points play in the properties of the spaces.

The space $H(K_p)$ with orthonormal basis $\{f_n\}_{n=0}^{\infty}$ where $f_n(z) = (1 - (\frac{n+1}{n+2})^p z) z^n$ is the main focus of study in Adams and McGuire [3]. A straightforward argument verifies that the reproducing kernel K_p has domain $\mathbb{D} \cup \{1\}$. Since the additional point in the domain of K_p is on $\partial\mathbb{D}$, this space inherits some interesting structure. The next result is proved in [3].

Theorem 3.1 (Adams-McGuire). *If $H(K_p)$ is the tridiagonal reproducing kernel Hilbert space with orthonormal basis $\{f_n\}_{n=0}^{\infty}$ where $f_n(z) = (1 - (\frac{n+1}{n+2})^p z)z^n$, then the following hold.*

1. *if $p > \frac{1}{2}$, then M_z is bounded on $H(K_p)$;*
2. *if $p > \frac{1}{2}$, then $H(K_p)$ contains the polynomials;*
3. *the space $H(K_p) \overset{s}{\subseteq} H^2(\mathbb{D})$ for all $p > 0$; and*
4. *The space $H(K_p)$ decomposes as follows.*
   (a) *If $p > \frac{1}{2}$, then $H(K_p) \overset{s}{=} (1-z)H^2(\mathbb{D}) + \mathbb{C}K_p(z,1)$.*
   (b) *If $p = \frac{1}{2}$, then $H(K_p) \overset{s}{=} (1-z)\mathcal{A}_p + \mathbb{C}K_p(z,1)$ where $\mathcal{A}_p$ is dense in $H^2(\mathbb{D})$, but not equal to $H^2(\mathbb{D})$.*
   (c) *If $0 < p < \frac{1}{2}$, then $H(K_p) \overset{s}{=} (1-z)\mathcal{A}_p + \mathbb{C}K_p(z,1)$ where $\mathcal{A}_p$ is the orthogonal complement in $H^2(\mathbb{D})$ of the function*

$$g_p(z) = \sum_{n=0}^{\infty} \left(1 - \left(\frac{n+1}{n+2}\right)^p\right)(n+2)^p\, z^n.$$

This section adds to the above result in Adams and McGuire [3] by showing that the spectrum of M_z is the closed unit disk $\overline{\mathbb{D}}$. We make use of the following result in Adams and McGuire [2] regarding the spectral radius of M_z on a space with kernel $K(z,w) = \sum_{n=0}^{\infty} f_n(z)\overline{f_n(w)}$ where $f_n(z) = (a_n + b_n z)z^n$.

Theorem 3.2 (Adams-McGuire). *If M_z is bounded with spectral radius $\rho(M_z)$, then $\rho(M_z) \leq \alpha$ where*

$$\alpha = \limsup_{k\to\infty} \left(\sup_{n\geq 0} \left| \frac{a_{n+1}}{a_{n+k}} \right| + 2 \sup_{n \geq 0} \left| \frac{c_{n,k}}{c_{n+k-1,1}} \right| \right)^{\frac{1}{k}}$$

and

$$c_{n,k} = \frac{b_n}{a_{n+k+1}} - \frac{a_n}{a_{n+k}} \frac{b_{n+k}}{a_{n+k+1}}.$$

Theorem 3.3. *If $H(K_p)$ is the tridiagonal reproducing kernel Hilbert space with orthonormal basis $\{f_n\}_{n=0}^{\infty}$ where $f_n(z) = (1 - (\frac{n+1}{n+2})^p z)z^n$, then the spectrum of M_z is the closed unit disk.*

Proof. If $\lambda \in \mathbb{D}$, then $(M_z^* - \lambda I)k_\lambda = 0$ which implies $\mathbb{D} \subseteq \sigma(M_z^*)$. Hence $\sigma(M_z)$, which is the complex conjugate of $\sigma(M_z^*)$, contains the closed unit disk. Apply theorem 3.2 with $a_n = 1$, $b_n = (\frac{n+1}{n+2})^p$, and $c_{n,k} = (\frac{n+1}{n+2})^p - (\frac{n+k+1}{n+k+2})^p$ to obtain an upper bound for the spectral radius of M_z which we compute as

$$\begin{aligned}
\alpha &= \lim_{k\to\infty}\left(1 + 2\sup_{n\geq 0}\left|\frac{(\frac{n+k+1}{n+k+2})^p - (\frac{n+1}{n+2})^p}{(\frac{n+k+1}{n+k+2})^p - (\frac{n+k}{n+k+1})^p}\right|\right)^{\frac{1}{k}} \\
&= \lim_{k\to\infty}\left(1 + 2\sup_{n\geq 0}\left|\frac{1 - (\frac{n+1}{n+2})^p(\frac{n+k+2}{n+k+1})^p}{1 - (\frac{n+k}{n+k+1})^p(\frac{n+k+2}{n+k+1})^p}\right|\right)^{\frac{1}{k}} \\
&= \lim_{k\to\infty}\left(1 + 2\sup_{n\geq 0}\left|\frac{1 - (1 - \frac{1}{n+2})^p(1 + \frac{1}{n+k+1})^p}{1 - (1 - \frac{1}{n+k+1})^p(1 + \frac{1}{n+k+1})^p}\right|\right)^{\frac{1}{k}} \\
&= \lim_{k\to\infty}\left(1 + 2\sup_{n\geq 0}\left(\frac{1 - (1 - \frac{k}{(n+2)(n+k+1)})^p}{1 - (1 - \frac{1}{(n+k+1)^2})^p}\right)\right)^{\frac{1}{k}}.
\end{aligned}$$

Let

$$R(n,k) = \left(\frac{1 - (1 - \frac{k}{(n+2)(n+k+1)})^p}{1 - (1 - \frac{1}{(n+k+1)^2})^p}\right).$$

Notice that the numerator and denominator of $R(n,k)$ both involve terms of the form $f(x) = (1-x)^p$ where $0 \leq x \leq 1/2$. The first order Taylor approximation for f at 0 gives the bounds

$$1 - px - \frac{1}{2}\left(\sup_{0\leq t_x\leq 1/2} f''(t_x)\right)x^2 \leq f(x) \leq 1 - px + \frac{1}{2}\left(\sup_{0\leq t_x\leq 1/2} f''(t_x)\right)x^2$$

where $0 \leq t_x \leq 1/2$.

Since $f''(x)$ is continuous when $0 \leq x \leq 1/2$, $\sup\limits_{0\leq t_x\leq 1/2} f''(t_x) \leq M$ for some constant M. Let k be large enough so that $\frac{M}{2(n+k+1)^2} \leq \frac{p}{2}$. Applying the above bounds to $R(n,k)$ gives

$$\begin{aligned} R(n,k) &\leq \frac{\frac{pk}{(n+2)(n+k+1)} + \frac{M}{2}\left(\frac{k}{(n+2)(n+k+1)}\right)^2}{\frac{p}{(n+k+1)^2} - \frac{M}{2}\left(\frac{1}{(n+k+1)}\right)^4} \\ &= \frac{\frac{pk(n+k+1)}{(n+2)} + \frac{Mk^2}{2(n+2)^2}}{p - \frac{M}{2(n+k+1)^2}} \\ &\leq \frac{\frac{pk(n+k+1)}{(n+2)} + \frac{Mk^2}{2(n+2)^2}}{\frac{p}{2}} \\ &\leq \frac{2k(n+k+1)}{n+2} + \frac{Mk^2}{p(n+2)^2}. \end{aligned}$$

Substituting this bound for $R(n,k)$, we obtain

$$\alpha \leq \lim_{k\to\infty}\left(1 + 2\sup_{n\geq 0}\left(\frac{2k(n+k+1)}{n+2} + \frac{Mk^2}{p(n+2)^2}\right)\right)^{\frac{1}{k}}.$$

Since for each $k > 1$, $\frac{2k(n+k+1)}{n+2}$ and $\frac{Mk^2}{p(n+2)^2}$ are both non-increasing in n, the supremum occurs when $n = 0$. Take the logarithm of both sides to compute the limit in k. Thus α is bounded above by

$$\lim_{k\to\infty}\left(1 + 2k(k+1) + \frac{Mk^2}{2p}\right)^{\frac{1}{k}} = 1.$$

Since $\mathbb{D} \subset \sigma(M_z)$, $\alpha \geq 1$. Hence $\alpha = 1$ and $\sigma(M_z)$ is the closed unit disk. $\square$

Chapter 4

Five Diagonal Reproducing Kernels

This section is focused on the study of the five diagonal (or two bandwidth) reproducing kernel Hilbert space $H(LL^*)$ with orthonormal basis $\{f_n\}_{n=0}^{\infty}$ where

$$f_n(z) = \left(1 - \left(\frac{n+1}{n+2}\right) z\right) \left(1 - \left(\frac{n+1}{n+2}\right) e^{i\theta_0} z\right) z^n.$$

We will often write

$$f_n(z) = \left(a_n + b_n z + c_n z^2\right) z^n$$

where

$$a_n = 1,\ \ b_n = \left(\frac{n+1}{n+2}\right)\left(1 + e^{i\theta_0}\right),\ \text{ and }\ c_n = \left(\frac{n+1}{n+2}\right)^2 e^{i\theta_0}.$$

This is a generalization of the example in section 3 to the five diagonal case. Only the parameter $p = 1$ is considered in order for the computations to be manageable. The next proposition shows that the domain of the functions in $H(LL^*)$ is $\mathbb{D} \cup \{1, e^{-i\theta_0}\}$.

Proposition 4.1. *If $H(LL^*)$ is the reproducing kernel Hilbert space with orthonormal basis $\{f_n\}_{n=0}^{\infty}$ where*

$$f_n(z) = \left(1 - \left(\frac{n+1}{n+2}\right) z\right) \left(1 - \left(\frac{n+1}{n+2}\right) e^{i\theta_0} z\right) z^n,$$

then the largest natural domain of $H(LL^*)$ is $\mathbb{D} \cup \{1, e^{-i\theta_0}\}$.

Proof. Let $Dom(LL^*)$ be the largest natural domain of $H(LL^*)$. By Aronszajn [4], $Dom(LL^*) = \{z \in \mathbb{C} : \sum_{n=0}^{\infty} |f_n(z)|^2 < \infty\}$. If $|z| < 1$, then

$$\sum_{n=0}^{\infty} |f_n(z)|^2 = \sum_{n=0}^{\infty} \left| \left(1 - \left(\frac{n+1}{n+2}\right) z\right) \left(1 - \left(\frac{n+1}{n+2}\right) e^{i\theta_0} z\right) z^n \right|^2 < 16 \sum_{n=0}^{\infty} |z|^{2n} < \infty.$$

Also, since $|1 - \frac{n+1}{n+2} e^{i\theta}| \leq 2$ and $1 - \frac{n+1}{n+2} = \frac{1}{n+2}$,

$$\sum_{n=0}^{\infty} |f_n(1)|^2 = \sum_{n=0}^{\infty} \left| \left(1 - \frac{n+1}{n+2}\right) \left(1 - \frac{n+1}{n+2} e^{i\theta_0}\right) \right|^2 < 4 \sum_{n=0}^{\infty} \left| \frac{1}{n+2} \right|^2 < \infty, \text{ and}$$

$$\sum_{n=0}^{\infty} |f_n(e^{-i\theta_0})|^2 = \sum_{n=0}^{\infty} \left| \left(1 - \frac{n+1}{n+2} e^{-i\theta_0}\right) \left(1 - \frac{n+1}{n+2}\right) \right|^2 < 4 \sum_{n=0}^{\infty} \left| \frac{1}{n+2} \right|^2 < \infty.$$

Hence $\mathbb{D} \cup \{1, e^{-i\theta_0}\} \subseteq Dom(LL^*)$.

If $z \notin \mathbb{D} \cup \{1, e^{-i\theta_0}\}$, then $|z| \geq 1$ and

$$\left(1 - \left(\frac{n+1}{n+2}\right) z\right) \left(1 - \left(\frac{n+1}{n+2}\right) e^{i\theta_0} z\right) \to (1 - z)\left(1 - e^{i\theta_0} z\right) \neq 0.$$

Hence

$$\sum_{n=0}^{\infty} |f_n(z)|^2 = \sum_{n=0}^{\infty} \left| \left(1 - \left(\frac{n+1}{n+2}\right) z\right) \left(1 - \left(\frac{n+1}{n+2}\right) e^{i\theta_0} z\right) z^n \right|^2 = \infty$$

and $Dom(LL^*) = \mathbb{D} \cup \{1, e^{-i\theta_0}\}$. □

The next result shows that the functions in $H(LL^*)$ are contained in $H^2(\mathbb{D})$ for general $\theta_0 \in [0, 2\pi)$.

Theorem 4.2. *If $H(LL^*)$ is the reproducing kernel Hilbert space with orthonormal basis $\{f_n\}_{n=0}^{\infty}$ where*

$$f_n(z) = \left(1 - \left(\frac{n+1}{n+2}\right) z\right) \left(1 - \left(\frac{n+1}{n+2}\right) e^{i\theta_0} z\right) z^n,$$

then $H(LL^*) \overset{s}{\subseteq} H^2(\mathbb{D})$ for each $\theta_0 \in [0, 2\pi)$.

Proof. Let $f \in H(LL^*)$. Then

$$f = \sum_{n=0}^{\infty} \alpha_n \left(1 - \left(\frac{n+1}{n+2}\right) z\right)\left(1 - \left(\frac{n+1}{n+2}\right) e^{i\theta_0} z\right) z^n$$

for some $\{\alpha_n\} \in \ell^2_+$. Expand this expression to write

$$f = \alpha_0 + \left(\alpha_1 - \frac{\alpha_0}{2}\left(1 + e^{i\theta_0}\right)\right) z + \sum_{n=2}^{\infty} \left(\alpha_n - \frac{n}{n+1}\alpha_{n-1}\left(1 + e^{i\theta_0}\right) + \left(\frac{n-1}{n}\right)^2 \alpha_{n-2} e^{i\theta_0}\right) z^n.$$

Since $\frac{n}{n+1} < 1$ and $\{\alpha_n\} \in \ell^2_+$,

$$\left\{\frac{n}{n+1}\left(1 + e^{i\theta_0}\right)\alpha_{n-1}\right\} \in \ell^2_+.$$

By the same reasoning,

$$\left\{\left(\frac{n-1}{n}\right)^2 \alpha_{n-2} e^{i\theta_0}\right\} \in \ell^2_+.$$

Thus

$$\left\{\alpha_n - \frac{n}{n+1}\alpha_{n-1}\left(1 + e^{i\theta_0}\right) + \left(\frac{n-1}{n}\right)^2 \alpha_{n-2} e^{i\theta_0}\right\} \in \ell^2_+$$

and hence $f \in H^2(\mathbb{D})$. ☐

Before focusing further on this example, a few general observations about five diagonal reproducing kernel Hilbert spaces are in order.

We begin by considering the relationship between $H(LL^*)$ and $H(\widehat{L}\widehat{L}^*)$ with

$$L = \begin{pmatrix} a_0 & 0 & \cdots & & & & \\ b_0 & a_1 & 0 & \cdots & & & \\ c_0 & b_1 & a_2 & 0 & \cdots & & \\ 0 & c_1 & b_2 & a_3 & 0 & \cdots & \\ 0 & 0 & c_2 & b_3 & a_4 & 0 & \cdots \\ & & & & \ddots & & \end{pmatrix}$$

and

$$\widehat{L} = \begin{pmatrix} \widehat{a}_0 & 0 & \cdots & & & & \\ \widehat{b}_0 & \widehat{a}_1 & 0 & \cdots & & & \\ \widehat{c}_0 & \widehat{b}_1 & \widehat{a}_2 & 0 & \cdots & & \\ 0 & \widehat{c}_1 & \widehat{b}_2 & \widehat{a}_3 & 0 & \cdots & \\ 0 & 0 & \widehat{c}_2 & \widehat{b}_3 & \widehat{a}_4 & 0 & \cdots \\ & & & & \ddots & & \end{pmatrix}$$

where the diagonal entries of L and $\widehat{L}$ are assumed to be nonzero. Suppose that the matrix $C = (c_{j,k})_{j,k=0}^{\infty}$ satisfies $\widehat{L} = LC$. Since L and $\widehat{L}$ have finite bandwidth, are lower triangular, and have nonzero diagonal entries, C is also a well defined lower triangular matrix with nonzero diagonal entries.

Remark (Range Inclusion)**.** In order to study the relationship between $H(LL^*)$ and $H(\widehat{L}\widehat{L}^*)$, we study properties of C. For example, applying Douglas' Lemma allows one to see that C is bounded if and only if $H(\widehat{L}\widehat{L}^*) \overset{s}{\subseteq} H(LL^*)$; C is invertible if and only if $H(LL^*)$ and $H(\widehat{L}\widehat{L}^*)$ contain the same functions; and the columns of C are bounded if and only if the columns of $\widehat{L}$ are the power series coefficients of functions in $H(LL^*)$. Note that if $\widehat{L}$ is the identity matrix, then $H(\widehat{L}\widehat{L}^*) = H(I)$ is the Hardy space $H^2(\mathbb{D})$. In particular, the existence of a bounded matrix C such that $I = \widehat{L} = LC$ is equivalent to the Hardy space $H^2(\mathbb{D})$ being contained in $H(LL^*)$. It is possible that $I = \widehat{L} = LC$ where C is not bounded. In that case we can still observe that the orthonormal basis vectors $\{e_n\}_{n=0}^{\infty}$ are in the range space of L if and only if the columns of C are vectors in ℓ^2_+. Thus the polynomials are in $H(LL^*)$ if and only if there is a C whose columns are vectors in ℓ^2_+ such that $I = \widehat{L} = LC$.

The entries of C are obtained from L and $\widehat{L}$ in a way that is similar to the three bandwidth case in Adams and McGuire [2]. However what is a scalar argument in [2] requires a matrix argument in the five bandwidth setting. We first need to introduce some notation and ultimately define a new matrix C_e, which is related to C, that will help us set up the matrix argument we require.

Suppose first that C is a matrix satisfying $\widehat{L} = LC$ where $\widehat{L} = [\widehat{\ell}_{j,k}]$ and $L = [\ell_{j,k}]$. Since L and $\widehat{L}$ are lower triangular, it follows that C is lower triangular.

If $0 \le k \le j$, then

$$\widehat{\ell}_{j,k} = \sum_{s=0}^{\infty} \ell_{j,s} c_{s,k}.$$

Since L is lower triangular, $\ell_{j,s} = 0$ if $s > j$. Therefore

$$\widehat{\ell}_{j,k} = \sum_{s=0}^{j} \ell_{j,s} c_{s,k}.$$

Since L has finite bandwidth, $\ell_{j,s} = 0$ if $s < j - 2$. So

$$\widehat{\ell}_{j,k} = \sum_{s=j-2}^{j} \ell_{j,s} c_{s,k}.$$

Using the specific form of L and being careful to note throughout the difference between the entries $c_j, \widehat{c}_j$ in the matrices L and $\widehat{L}$ and the entries $c_{j,k}$ in the matrix C, we obtain the equation

$$\widehat{\ell}_{j,k} = c_{j-2} c_{j-2,k} + b_{j-1} c_{j-1,k} + a_j c_{j,k}$$

where $c_{x,y} = 0$ if x happens to be negative.

Depending on the respective cases $j = k$, $j = k + 1$, $j = k + 2$, or $j > k + 2$, the equation

$$\widehat{\ell}_{j,k} = \sum_{s=j-2}^{j} \ell_{j,s} c_{s,k}$$

leads to the equations

$$\begin{aligned} \widehat{a}_j = \widehat{\ell}_{j,j} &= a_j c_{j,j}, && (4.1)\\ \widehat{b}_{j-1} = \widehat{\ell}_{j,j-1} &= b_{j-1} c_{j-1,j-1} + a_j c_{j,j-1} && (4.2)\\ \widehat{c}_{j-2} = \widehat{\ell}_{j,j-2} &= c_{j-2} c_{j-2,j-2} + b_{j-1} c_{j-1,j-2} + a_j c_{j,j-2} && (4.3)\\ 0 = \widehat{\ell}_{j,k} &= c_{j-2} c_{j-2,k} + b_{j-1} c_{j-1,k} + a_j c_{j,k} && (4.4) \end{aligned}$$

Notice that equation (4) can be conveniently expressed using vector notation. To this end, let $\vec{v}_{j,k} = \begin{pmatrix} c_{j-1,k} \\ c_{j,k} \end{pmatrix}$ where again $c_{-1,k} = 0$ and let

$$M_n = \begin{pmatrix} 0 & 1 \\ -\frac{c_{n-2}}{a_n} & -\frac{b_{n-1}}{a_n} \end{pmatrix}.$$

Equation (4) is now encoded by the equation

$$\vec{v}_{j,k} = M_j \vec{v}_{j-1,k}$$

where $j > k+2$, since

$$\vec{v}_{j,k} = \begin{pmatrix} c_{j-1,k} \\ c_{j,k} \end{pmatrix} = \begin{pmatrix} c_{j-1,k} \\ -\frac{c_{j-2}}{a_j} c_{j-2,k} - \frac{b_{j-1}}{a_j} c_{j-1,k} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -\frac{c_{n-2}}{a_n} & -\frac{b_{n-1}}{a_n} \end{pmatrix} \begin{pmatrix} c_{j-2,k} \\ c_{j-1,k} \end{pmatrix} = M_j \vec{v}_{j-1,k}.$$

Moreover, if $j > k+2$, then

$$(*) \quad \vec{v}_{j,k} = M_j \vec{v}_{j-1,k} = M_j M_{j-1} \vec{v}_{j-2,k} = \cdots = M_j M_{j-1} \cdots M_{k+3} \vec{v}_{k+2,k}.$$

The recursion suggests we introduce the expanded matrix C_e of C defined by

$$C_e = \begin{pmatrix} 0 & 0 & \cdots & & \\ c_{0,0} & 0 & \cdots & & \\ c_{0,0} & 0 & \cdots & & \\ c_{1,0} & c_{1,1} & 0 & \cdots & \\ c_{1,0} & c_{1,1} & 0 & \cdots & \\ c_{2,0} & c_{2,1} & c_{2,2} & 0 & \cdots \\ c_{2,0} & c_{2,1} & c_{2,2} & 0 & \cdots \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix} = \begin{pmatrix} \vec{v}_{0,0} & \vec{0} & \vec{0} & \cdots \\ \vec{v}_{1,0} & \vec{v}_{1,1} & \vec{0} & \cdots \\ \vec{v}_{2,0} & \vec{v}_{2,1} & \vec{v}_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

With the above recursion, notice

$$C_e = \begin{pmatrix} \vec{v}_{0,0} & \vec{0} & \cdots & & & & & \\ \vec{v}_{1,0} & \vec{v}_{1,1} & \vec{0} & \cdots & & & & \\ \vec{v}_{2,0} & \vec{v}_{2,1} & \vec{v}_{2,2} & \vec{0} & \cdots & & & \\ M_3\vec{v}_{2,0} & \vec{v}_{3,1} & \vec{v}_{3,2} & \vec{v}_{3,3} & \vec{0} & \cdots & & \\ M_4M_3\vec{v}_{2,0} & M_4\vec{v}_{3,1} & \vec{v}_{4,2} & \vec{v}_{4,3} & \vec{v}_{4,4} & \vec{0} & \cdots & \\ M_5M_4M_3\vec{v}_{2,0} & M_5M_4\vec{v}_{3,1} & M_5\vec{v}_{4,2} & \vec{v}_{5,3} & \vec{v}_{5,4} & \vec{v}_{5,5} & \vec{0} & \cdots \\ \vdots & \vdots & \vdots & & \ddots & & & \end{pmatrix}.$$

Notice that by the Range Inclusion Remark 4, C is bounded if and only if C_e is bounded, and the columns of C are elements of ℓ^2_+ if and only if the columns of C_e are elements of ℓ^2_+. The matrix C_e allows for the easy observation of the recursion present in the entries of C. Therefore, it is convenient to consider C_e in lieu of C.

We now return to consideration of the space $H(LL^*)$ with orthonormal basis $\{f_n\}_{n=0}^{\infty}$ where $f_n(z) = (1 - (\frac{n+1}{n+2})z)(1 - (\frac{n+1}{n+2})e^{i\theta_0}z)z^n$. We will show that when $\theta_0 = \pi$ and $\theta_0 = \frac{2\pi}{3}$, the polynomials are contained in $H(LL^*)$. Additionally, we will show that when $\theta_0 = \pi$, not all of the functions in $H^2(\mathbb{D})$ are contained in $H(LL^*)$.

We first treat the case where $\theta_0 = \pi$. Our immediate goal is to show that $H(LL^*)$ contains the polynomials. In order to establish this result, we will let $\widehat{L}$ be the identity matrix and show that the columns of C_e are vectors in ℓ^2_+.

Notice for this example, $M_n = \begin{pmatrix} 0 & 1 \\ (\frac{n-1}{n})^2 & 0 \end{pmatrix}$. In order to simplify this argument, let $W_n = M_n M_{n-1} \cdots M_3$. Note that if $\displaystyle\sum_{n=3}^{\infty} ||W_n||^2 < \infty$, then $\displaystyle\sum_{n=k}^{\infty} ||M_n M_{n-1} \cdots M_k||^2 < \infty$ for each $k \geq 3$. To see this, observe that M_j is invertible for each $j \geq 3$ and $M_n M_{n-1} \cdots M_k = W_n M_3^{-1} M_4^{-1} \cdots M_{k-1}^{-1}$. Thus

$$\sum_{n=k}^{\infty} \|M_n M_{n-1} \cdots M_k\|^2 \leq \|M_3^{-1} M_4^{-1} \cdots M_{k-1}^{-1}\|^2 \sum_{n=k}^{\infty} \|W_n\|^2 < \infty.$$

Notice

$$\begin{aligned} \vec{v}_{j,k} &= M_j M_{j-1} \cdots M_{k+3} \vec{v}_{k+2,k} \\ &= M_j M_{j-1} \cdots M_{k+3} M_{k+2} M_{k+1} M_k M_k^{-1} M_{k+1}^{-1} M_{k+2}^{-1} \vec{v}_{k+2,k} \end{aligned}$$

for $j \geq k+3$. So if $\displaystyle\sum_{n=k}^{\infty} ||M_n M_{n-1} \cdots M_k||^2 < \infty$, then

$$\sum_{j=k}^{\infty} \|\vec{v}_{j,k}\|^2 \leq \|M_k^{-1} M_{k+1}^{-1} M_{k+2}^{-1} \vec{v}_{k+2,k}\|^2 \sum_{j=k}^{\infty} \|M_j M_{j-1} \cdots M_k\|^2 < \infty.$$

Hence the k^{th} column of C_e is in ℓ^2_+. Therefore, it suffices to show $\displaystyle\sum_{n=3}^{\infty} ||W_n||^2 < \infty$. Lemma 4.3 gives W_n explicitly.

Lemma 4.3. *Let $\theta_0 = \pi$ and $x_n = (\frac{n-1}{n})^2$. If $H(LL^*)$ is the reproducing kernel Hilbert space with orthonormal basis $\{f_n\}_{n=0}^{\infty}$ where*

$$f_n(z) = \left(1 - \left(\frac{n+1}{n+2}\right) z\right) \left(1 - \left(\frac{n+1}{n+2}\right) e^{i\theta_0} z\right) z^n, \text{ then}$$

$$W_n = \begin{pmatrix} 0 & \prod_{k=2}^{\frac{n-1}{2}} x_{2k} \\ \prod_{k=1}^{\frac{n-1}{2}} x_{2k+1} & 0 \end{pmatrix} \text{ for odd } n \text{ and } W_n = \begin{pmatrix} \prod_{k=1}^{\frac{n}{2}-1} x_{2k+1} & 0 \\ 0 & \prod_{k=2}^{\frac{n}{2}} x_{2k} \end{pmatrix} \text{ for even } n.$$

Proof. We will use induction on n and will first consider the case where n is odd. The base case holds true since

$$W_3 = M_3 = \begin{pmatrix} 0 & 1 \\ x_3 & 0 \end{pmatrix}.$$

Suppose the claim holds true for n. Then

$$W_{n+2} = M_{n+2}M_{n+1}W_n = \begin{pmatrix} 0 & 1 \\ x_{n+2} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ x_{n+1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \prod_{k=2}^{\frac{n-1}{2}} x_{2k} \\ \prod_{k=1}^{\frac{n-1}{2}} x_{2k+1} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \prod_{k=2}^{\frac{n+1}{2}} x_{2k} \\ \prod_{k=1}^{\frac{n+1}{2}} x_{2k+1} & 0 \end{pmatrix}.$$

Now consider the n even case. The base case holds true since

$$W_4 = M_4M_3 = \begin{pmatrix} 0 & 1 \\ x_4 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ x_3 & 0 \end{pmatrix} = \begin{pmatrix} x_3 & 0 \\ 0 & x_4 \end{pmatrix}.$$

Suppose the claim holds true for n. Then

$$W_{n+2} = M_{n+2}M_{n+1}W_n = \begin{pmatrix} 0 & 1 \\ x_{n+2} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ x_{n+1} & 0 \end{pmatrix} \begin{pmatrix} \prod_{k=1}^{\frac{n}{2}-1} x_{2k+1} & 0 \\ 0 & \prod_{k=2}^{\frac{n}{2}} x_{2k} \end{pmatrix}$$

$$= \begin{pmatrix} \prod_{k=1}^{\frac{n}{2}} x_{2k+1} & 0 \\ 0 & \prod_{k=2}^{\frac{n}{2}+1} x_{2k} \end{pmatrix}.$$

□

Theorem 4.4. *Let $\theta_0 = \pi$ and $x_n = (\frac{n-1}{n})^2$. If $H(LL^*)$ is the reproducing kernel Hilbert space with orthonormal basis $\{f_n\}_{n=0}^{\infty}$ where*

$$f_n(z) = \left(1 - \left(\frac{n+1}{n+2}\right) z\right)\left(1 - \left(\frac{n+1}{n+2}\right) e^{i\theta_0} z\right) z^n,$$

then $H(LL^*)$ contains the polynomials.

Proof. Recall from the Range Inclusion Remark 4 that $H(LL^*)$ contains the polynomials if and only if the columns of C_e are in ℓ^2_+. Therefore if $\sum_{n=3}^{\infty} ||W_n||^2 < \infty$, then $H(LL^*)$ contains the polynomials. For all n,

$$||W_n|| \leq \max\left\{\prod_{k=1}^{\frac{n}{2}-1} x_{2k+1}, \prod_{k=2}^{\frac{n}{2}} x_{2k}, \prod_{k=2}^{\frac{n-1}{2}} x_{2k}, \prod_{k=1}^{\frac{n-1}{2}} x_{2k+1}\right\}.$$

Observe that

$$\begin{aligned}
\prod_{k=1}^{\frac{n}{2}-1} x_{2k+1} &= \left(\frac{2}{3}\right)^2\left(\frac{4}{5}\right)^2 \cdots \left(\frac{n-2}{n-1}\right)^2 \leq \left(\frac{2}{3}\right)\left(\frac{3}{4}\right)\cdots\left(\frac{n-2}{n-1}\right)\left(\frac{n-1}{n}\right) = \frac{2}{n}, \\
\prod_{k=2}^{\frac{n}{2}} x_{2k} &= \left(\frac{3}{4}\right)^2\left(\frac{5}{6}\right)^2 \cdots \left(\frac{n-1}{n}\right)^2 \leq \left(\frac{3}{4}\right)\left(\frac{4}{5}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{n}{n+1}\right) = \frac{3}{n+1}, \\
\prod_{k=2}^{\frac{n-1}{2}} x_{2k} &= \left(\frac{3}{4}\right)^2\left(\frac{5}{6}\right)^2 \cdots \left(\frac{n-2}{n-1}\right)^2 \leq \left(\frac{3}{4}\right)\left(\frac{4}{5}\right)\cdots\left(\frac{n-2}{n-1}\right)\left(\frac{n-1}{n}\right) = \frac{3}{n}, \\
\prod_{k=1}^{\frac{n-1}{2}} x_{2k+1} &= \left(\frac{2}{3}\right)^2\left(\frac{4}{5}\right)^2 \cdots \left(\frac{n-1}{n}\right)^2 \leq \left(\frac{2}{3}\right)\left(\frac{3}{4}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{n}{n+1}\right) = \frac{2}{n+1}.
\end{aligned}$$

Thus $||W_n|| \leq \frac{3}{n}$ for any n. So $\sum_{n=1}^{\infty} ||W_n||^2 \leq \sum_{n=1}^{\infty} \left(\frac{3}{n}\right)^2 < \infty$ and $H(LL^*)$ contains the polynomials. □

We will now treat the case where $\theta_0 = 2\pi/3$. Our goal with this case is to show that $H(LL^*)$ contains the polynomials. In order to establish this result, we will let $\widehat{L}$ be the identity matrix and show the columns of C_e are bounded. Unfortunately, in this case, the product $M_n M_{n-1} \cdots M_3$ does not take such a nice form as the $\theta_0 = \pi$ case. We therefore employ a different approach to show the columns of C_e are bounded. Specifically, we show that, for each k, the sequence $\{\|\vec{v}_{j,k}\|\}$ is square summable in j. Notice

$$\sum_{j=0}^{\infty} \|\vec{v}_{j,k}\|^2 = \sum_{j=0}^{\infty} \|\vec{v}_{3j,k}\|^2 + \sum_{j=0}^{\infty} \|\vec{v}_{3j+1,k}\|^2 + \sum_{j=0}^{\infty} \|\vec{v}_{3j+2,k}\|^2.$$

We will show $\{\|\vec{v}_{j,k}\|\}$ is square summable in j by proving the three subsequences $\{\|\vec{v}_{3j,k}\|\}, \{\|\vec{v}_{3j+1,k}\|\},$ and$\{\|\vec{v}_{3j+2,k}\|\}$ are square summable in j.

Theorem 4.5. *Let $\theta_0 = \frac{2\pi}{3}$. If $H(LL^*)$ is the reproducing kernel Hilbert space with orthonormal basis $\{f_n\}_{n=0}^{\infty}$ where*

$$f_n(z) = \left(1 - \left(\frac{n+1}{n+2}\right) z\right)\left(1 - \left(\frac{n+1}{n+2}\right) e^{i\theta_0} z\right) z^n,$$

then $H(LL^*)$ contains the polynomials.

Proof. We will prove that $\{\|\vec{v}_{3j,k}\|\}, \{\|\vec{v}_{3j+1,k}\|\},$ and$\{\|\vec{v}_{3j+2,k}\|\}$ are square summable in j. Notice that

$$M_n = \begin{pmatrix} 0 & 1 \\ -(\frac{n-1}{n})^2 e^{\frac{2\pi}{3}i} & \frac{n}{n+1}(1 + e^{\frac{2\pi}{3}i}) \end{pmatrix}.$$

Compute to see

$$M_{n+1} M_n M_{n-1} = \begin{pmatrix} \frac{n(n-2)^2}{(n+1)(n-1)^2} & e^{\frac{2\pi}{3}i}\frac{n-1}{n^2(n+1)} \\ (1 + e^{\frac{2\pi}{3}i})\frac{n(n-2)^2}{(n+2)(n^2-1)^2} & \frac{n^5+n^4-3n^3-n^2+n+1}{n^2(n+1)^2(n+2)} \end{pmatrix}$$

$$= \left(1 - \frac{3}{n+1}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{n+1}\begin{pmatrix} \frac{2-n}{(n-1)^2} & e^{\frac{2\pi}{3}i}\frac{n-1}{n^2} \\ (1 + e^{\frac{2\pi}{3}i})\frac{n(n-2)^2}{(n+1)(n+2)(n-1)^2} & \frac{n^3+3n^2+n+1}{n^2(n+1)(n+2)}) \end{pmatrix}.$$

Let

$$r_n = \left\| \begin{pmatrix} \frac{2-n}{(n-1)^2} & e^{\frac{2\pi}{3}i}\frac{n-1}{n^2} \\ (1 + e^{\frac{2\pi}{3}i})\frac{n(n-2)^2}{(n+1)(n+2)(n-1)^2} & \frac{n^3+3n^2+n+1}{n^2(n+1)(n+2)}) \end{pmatrix} \right\|.$$

Notice $\lim_{n\to\infty} r_n = 0$, so there exists an $N_0 > 0$ such that if $n > N_0$, $r_n \leq 1$. Let $N = \max\{N_0, k+2\}$. So if $n > N$, $||M_{n+1}M_nM_{n-1}|| \leq (1 - \frac{3}{n+1} + \frac{r_n}{n+1}) \leq (1 - \frac{2}{n+1})$.

Let

$$W_{0,j} = M_{N+3j}M_{N+3j-1}M_{N+3j-2},$$
$$W_{1,j} = M_{N+3j+1}M_{N+3j}M_{N+3j-1}, \quad \text{and}$$
$$W_{2,j} = M_{N+3j+2}M_{N+3j+1}M_{N+3j}.$$

Recall N is chosen so that $N \geq k+2$. The application of recursion $(*)$ relating to the columns of C_e yields $\vec{v}_{N+1,k} = M_{N+1}\vec{v}_{N,k}$. Therefore, we have

$$\vec{v}_{N+3j,k} = W_{0,j}W_{0,j-1}\cdots W_{0,1}\vec{v}_{N,k},$$
$$\vec{v}_{N+3j+1,k} = W_{1,j}W_{1,j-2}\cdots W_{1,1}\vec{v}_{N+1,k}, \quad \text{and}$$
$$\vec{v}_{N+3j+2,k} = W_{2,j}W_{2,j-1}\cdots W_{2,1}\vec{v}_{N+2,k}.$$

Thus, to show $\{\|\vec{v}_{3j,k}\|\}, \{\|\vec{v}_{3j+1,k}\|\}, \{\|\vec{v}_{3j+2,k}\|\}$ are square summable in j, it suffices to show

$$\sum_{j=1}^{\infty} ||W_{0,j}W_{0,j-1}\cdots W_{0,1}||^2 < \infty,$$
$$\sum_{j=1}^{\infty} ||W_{1,j}W_{1,j-1}\cdots W_{1,1}||^2 < \infty, \quad \text{and}$$
$$\sum_{j=1}^{\infty} ||W_{2,j}W_{2,j-1}\cdots W_{2,1}||^2 < \infty \text{ respectively.}$$

Since $||M_{n+1}M_nM_{n-1}|| \leq (1 - \frac{2}{n+1})$ for all $n > N$,

$$\|W_{0,j}\| \leq (1-\frac{2}{N+3j}), \quad \|W_{1,j}\| \leq (1-\frac{2}{N+3j+1}), \quad \text{and} \quad \|W_{2,j}\| \leq (1-\frac{2}{N+3j+2}).$$

Thus $\|W_{0,j}\|, \|W_{1,j}\|, \|W_{2,j}\| \leq (1 - \frac{2}{N+3j+2})$. We will complete the proof by showing $\sum_{j=1}^{\infty}\prod_{m=1}^{j}\left(1 - \frac{2}{N+3m+2}\right)^2 < \infty$.

Let

$$y = \log \prod_{m=1}^{j}\left(1 - \frac{2}{N+3m+2}\right) = \sum_{m=1}^{j} \log\left(1 - \frac{2}{N+3m+2}\right).$$

Since $\log(1-x) \leq -x$ for $x \in (-\infty, 1)$,

$$y \leq \sum_{m=1}^{j} \frac{-2}{N+3m+2}.$$

Let $f(x) = \frac{2}{N+3x+2}$. Since f decreases as x increases,

$$\sum_{m=1}^{j} \frac{2}{N+3m+2} \geq \int_{1}^{j+1} \frac{2}{N+3x+2} dx = -\log\left(\frac{N+5}{N+3j+5}\right)^{2/3}.$$

Therefore,

$$y \leq \sum_{m=1}^{j} \frac{-2}{N+3m+2} \leq \int_{1}^{j+1} \frac{-2}{N+3x+2} dx = \log\left(\frac{N+5}{N+3j+5}\right)^{2/3}.$$

So

$$\prod_{m=1}^{j}\left(1 - \frac{2}{N+3m+2}\right) = e^{y} \leq \left(\frac{N+5}{N+3j+5}\right)^{2/3}.$$

Thus

$$\sum_{j=1}^{\infty}\left(\prod_{m=1}^{j}\left(1 - \frac{2}{N+3m+2}\right)\right)^{2} \leq \sum_{j=1}^{\infty}\left(\frac{N+5}{N+3j+5}\right)^{4/3} < \infty.$$

□

The next result shows that when $\theta_0 = \pi$, not all of the functions in $H^2(\mathbb{D})$ are contained in $H(LL^*)$. As we will see, this argument generalizes to show that $H^2(\mathbb{D}) \overset{s}{\nsubseteq} H(LL^*)$ whenever $\theta_0 \in (0, 2\pi)$,

Theorem 4.6. *Let $\theta_0 = \pi$. If $H(LL^*)$ is the reproducing kernel Hilbert space with orthonormal basis $\{f_n\}_{n=0}^{\infty}$ where*

$$f_n(z) = \left(1 - \left(\frac{n+1}{n+2}\right) z\right)\left(1 - \left(\frac{n+1}{n+2}\right) e^{i\theta_0} z\right) z^n,$$

then $H^2(\mathbb{D}) \overset{s}{\nsubseteq} H(LL^*)$.

Proof. By the Range Inclusion Remark 4, $H^2(\mathbb{D}) \overset{s}{\subseteq} H(LL^*)$ if and only if $I = LC$ for some bounded C. Notice that such a C would be a right inverse of L, so to prove $H^2(\mathbb{D}) \overset{s}{\nsubseteq} H(LL^*)$, we show no such bounded right inverse exists. Notice that

$$L = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ -\frac{1}{4} & 0 & 1 & \cdots \\ 0 & -\frac{4}{9} & 0 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

If $LC = I$, we see that $C = \{c_{j,k}\}_{j,k=0}^{\infty}$ with

$$c_{j,k} = \begin{cases} 0 & \text{if } j < k \text{ or } j+k \equiv 1 \mod 2 \\ 1 & \text{if } j = k \\ (\frac{j-1}{j})^2 c_{j-2,k} & \text{otherwise} \end{cases}.$$

Explicitly,

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ (\frac{1}{2})^2 & 0 & 1 & 0 & \cdots \\ 0 & (\frac{2}{3})^2 & 0 & 1 & \cdots \\ (\frac{1}{2})^2(\frac{3}{4})^2 & 0 & (\frac{3}{4})^2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $K \in \mathbb{Z}^+$ be given. Choose $j \in \mathbb{Z}^+$ such that $(\frac{j+1}{j+2})^{2K} > \frac{1}{2}$. Notice that the first K nonzero terms of Ce_j (the j^{th} column of C) are $1, (\frac{j+1}{j+2})^2, (\frac{j+1}{j+2})^2(\frac{j+3}{j+4})^2, \ldots,$ and $(\frac{j+1}{j+2})^2(\frac{j+3}{j+4})^2 \cdots (\frac{j+2K-3}{j+2K-2})^2$. Since each of the first K nonzero terms of Ce_j are greater than $(\frac{j+1}{j+2})^{2K}$, our assumption on j guarantees each of the first K nonzero terms of Ce_j are greater than $\frac{1}{2}$. Therefore, $||Ce_j||^2 > \frac{1}{4}K$. Hence $\sup_{j \in \mathbb{Z}^+} ||Ce_j|| = \infty$, and so C is not bounded. We conclude that $H^2(\mathbb{D}) \overset{s}{\not\subseteq} H(LL^*)$. $\square$

The argument above which shows that $H^2(\mathbb{D})$ is not contained in $H(LL^*)$ for $\theta_0 = \pi$ generalizes for all $\theta_0 \in (0, 2\pi)$. In that case, we effectively wrote

$$M_n = M_\infty + \frac{1}{n} E_n$$

where

$$M_\infty = \begin{pmatrix} 0 & 1 \\ -e^{i\theta_0} & 1 + e^{i\theta_0} \end{pmatrix} \text{ and } \sup \|E_n\| < \infty.$$

When $\theta_0 = \pi$, $M_\infty = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The argument above relied on the fact that, for each $K \in \mathbb{Z}^+$, there exists an N such that if $n \geq N$ and $k \leq K$, then

$$\|\vec{v}_{n+k,n}\| = \|M_{n+k} M_{n+k-1} \cdots M_{n+3} \vec{v}_{n+2,n}\| \approx \|M_\infty^k \vec{v}_{n+2,n}\|.$$

For $\theta_0 = \pi$, M_∞ has eigenvalues 1 and -1, and $\|M_\infty^k \vec{v}_{n+2,n}\| = \|\vec{v}_{n+2,n}\|$. For general $\theta_0 \in (0, 2\pi)$, M_∞ has eigenvalues 1 and $e^{i\theta_0}$, so this must be slightly modified and we find a constant δ such that

$$\|M_\infty^k \vec{v}_{n+2,n}\| \geq \delta \|\vec{v}_{n+2,n}\|$$

for $0 \leq k \leq K$ and $n \geq N$. This shows that the ℓ^2_+ norm of the n^{th} column of C_e exceeds $\sqrt{K}\delta\|\vec{v}_{n+2,n}\|$. A computation based on the four recursion equations from this section shows that for large n,

$$\vec{v}_{n+2,n} \approx \begin{pmatrix} 1+e^{i\theta_0} \\ 1+e^{i\theta_0}+e^{2i\theta_0} \end{pmatrix}.$$

Thus the ℓ^2_+ norm of the n^{th} column of C_e exceeds $d\sqrt{K}\delta$ where $d = \left\| \begin{pmatrix} 1+e^{i\theta_0} \\ 1+e^{i\theta_0}+e^{2i\theta_0} \end{pmatrix} \right\|$. In particular, the ℓ^2_+ norms of the columns of C_e are not bounded. This leads to the following theorem.

Theorem 4.7. *Let $\theta_0 \in (0, 2\pi)$. If $H(LL^*)$ is the reproducing kernel Hilbert space with orthonormal basis $\{f_n\}_{n=0}^{\infty}$ where*

$$f_n(z) = \left(1 - \left(\frac{n+1}{n+2}\right) z\right)\left(1 - \left(\frac{n+1}{n+2}\right) e^{i\theta_0} z\right) z^n,$$

then $H^2(\mathbb{D}) \overset{s}{\not\subseteq} H(LL^*)$.

Chapter 5

Summative Conclusion

The motivation for the work in this thesis was to generalize the study of diagonal reproducing kernel Hilbert spaces to the bandwidth J case. This was first done by Adams and McGuire in [2]. One particularly interesting result about the bandwidth J spaces is that the natural domain of the functions in such a space is an open or closed disk together with at most J additional points. The tridiagonal space $H(K_p)$ with orthonormal basis $\{f_n\}_{n=0}^{\infty}$ where

$$f_n(z) = \left(1 - \left(\frac{n+1}{n+2}\right)^p z\right) z^n$$

was studied in [2] and [3]. In this space, an additional domain point is added on the boundary of the unit disk, leading to distinctly different behavior from the Hardy space or any diagonal space. Theorem 3.3 furthers the study of this space $H(K_p)$ by proving the spectrum of M_z is the closed unit disk.

The main goal of this thesis is to understand the structure of higher bandwidth spaces and their multiplication operators, in order to generalize the results of the tridiagonal case. Section 4 uses Douglas' Lemma to provide a framework for analyzing these higher bandwidth spaces. In particular, we can study the relationship between two spaces $H(\widehat{L}\widehat{L}^*)$ and $H(LL^*)$ by examining the matrix C where $\widehat{L} = LC$. If $H(\widehat{L}\widehat{L}^*)$ and $H(LL^*)$ are five diagonal spaces, some essential properties of the matrix C hold if and only if those same properties hold for an expanded matrix C_e. In this case, C_e takes a nice form where the columns are given by a recursion involving 2 by 2 matrices.

The five diagonal space $H(LL^*)$ with orthonormal basis $\{f_n\}_{n=0}^{\infty}$ where

$$f_n(z) = \left(1 - \left(\frac{n+1}{n+2}\right) z\right) \left(1 - \left(\frac{n+1}{n+2}\right) e^{i\theta_0} z\right) z^n$$

generalizes the tridiagonal example of [2] and [3] so that two additional domain points are now included on the boundary of the unit disk. The framework of section 4 is applied to this example to show that $H(LL^*)$ properly contains the polynomials and is properly contained in $H^2(\mathbb{D})$. Furthermore, this framework lays the foundation for future work in determining the multiplier algebra, answering the open questions in section 6, and investigating the general structure of higher bandwidth spaces.

Chapter 6

Open Questions

The following questions pertain to the space $H(K)$ studied in section 4.

Question 6.1. It is reasonable to expect that $H(K)$ behaves somewhat analogously to the three-diagonal space studied in Example 3.1 motivating the following question.

Are the functions in $H(K)$ all of the form

$$(z-1)(z-e^{-i\theta})g + \alpha K_1(z) + \beta\left[K_{e^{i\theta}}(z) - \frac{K(1,e^{i\theta})}{K(1,1)}K_1(z)\right]$$

where $\alpha, \beta \in \mathbb{C}$ and $g \in H^2(\mathbb{D})$? Note that the last two terms simply describe the span of the two vectors K_1 and $K_{e^{i\theta}}$.

Question 6.2. Does $H(K)$ contain the polynomials for all $\theta_0 \in [0, 2\pi)$?

It is expected that the methods used in Theorem 4.5 generalize to answer this question affirmatively where θ_0 is a rational multiple of π. The answer to this question for θ_0 not of this form will likely require a somewhat different approach.

The following questions relating to multiplication operators on $H(K)$ were all motivating questions for this thesis and time did not permit there consideration. It is hoped that the framework developed in this thesis will be helpful in answering them.

Question 6.3. What are the multipliers of $H(K)$? Equivalently, which multiplication operators M_ϕ are bounded?

Question 6.4. Is z a multiplier of $H(K)$? Equivalently, is M_z bounded on $H(K)$?

Question 6.5. Assuming that M_z is bounded on $H(K)$.

1. What is the spectrum of M_z?
2. What is the spectral radius of M_z?
3. What is the norm of M_z?
4. Is M_z similar to a rank two perturbation of the unilateral shift S on $H^2(\mathbb{D})$?

Question 6.6. How does the example in section 4 generalize if the orthonormal basis $\{f_n\}_{n=0}^{\infty}$ is changed to

$$f_n(z) = \prod_{m=1}^{J} \left(1 - \left(\frac{n+1}{n+2}\right)^p e^{i\theta_m} z\right) z^n \,?$$

What role does $p > 0$ play? Are there intervals of p which separate the behavior of $H(K)$ as in Example 3.1?

One could first consider the simplest case $p = 1$ in which case the orthonormal basis $\{f_n\}_{n=0}^{\infty}$ is given by

$$f_n(z) = \prod_{m=1}^{J} \left(1 - \left(\frac{n+1}{n+2}\right) e^{i\theta_m} z\right) z^n.$$

Chapter 7

Appendix

The following are basic results that can be found in any functional analysis text (see [5]).

Theorem 7.1 (Cantor Intersection Theorem)**.** *If X is a complete metric space, $\{F_n\}$ is a sequence of nonempty closed sets such that $F_{n+1} \subset F_n$ for all n, and $d_n \to 0$ where d_n is the diameter of F_n, then $\bigcap F_n$ is nonempty.*

Proof. For each n, let $x_n \in F_n$. Since $d_n \to 0$ and $F_{n+1} \subset F_n$ for all n, $\{x_n\}$ is a Cauchy sequence. Since X is complete, $\{x_n\}$ converges to some $x \in X$. Since $\{x_k\}_{k \geq n} \subset F_n$ for all $n \in \mathbb{N}$, x is a limit point of each F_n. Since each F_n is closed, $x \in F_n$ for each n. Thus $x \in \bigcap F_n$. $\square$

Definition 7.2 (Baire Space)**.** A Baire space X is a topological space such that whenever $\{O_n\}$ is a countable collection of open dense sets in X, the intersection $\bigcap O_n$ is a dense set.

Theorem 7.3. *Every complete metric space X is a Baire space.*

Proof. Let $\{O_n\}$ be a sequence of dense open sets in X and let $O \subseteq X$ be an arbitrary open set. Since O_1 is dense and open, $O_1 \cap O$ has nonempty interior. Let $x_1 \in \text{int}(O_1 \cap O)$. Hence there is a closed disk $\overline{B(x_1, r_1)}$ of radius r_1 about x_1 entirely contained in $O_1 \cap O$. For the same reason, $O_2 \cap B(x_1, r_1)$ contains a closed disc $\overline{B(x_2, r_2)}$ where $x_2 \in \text{int}(O_2 \cap B(x_1, r_1))$ and $0 < r_2 < \frac{1}{2} r_1$. Continuing in this way, we obtain a nested sequence $\{\overline{B(x_n, r_n)}\}$ of closed disks whose diameters $r_n \to 0$. By the Cantor Intersection Theorem,

$$O \cap \left(\bigcap_{n=1}^{\infty} O_n \right) = (O \cap O_1) \cap \left(\bigcap_{n=2}^{\infty} O_n \right) \supseteq \bigcap_{n=1}^{\infty} \overline{B(x_n, r_n)} \neq \emptyset.$$

Thus $O \cap \bigcap_{n=1}^{\infty} O_n \neq \emptyset$, and so $\bigcap_{n=1}^{\infty} O_n$ is a dense set. □

Definition 7.4. Let S be a subset of a topological space X.

1. The set S is nowhere dense in X if the closure of S has empty interior.
2. The set S is of first category in X if S is the countable union of nowhere dense sets in X.
3. The set S is of second category in X if it is not of first category in X.

Example 7.5. The set of rational numbers $\mathbb{Q}$ is of first category in $\mathbb{R}$ since it is the countable union of singleton sets, each of which is nowhere dense in $\mathbb{R}$.

Theorem 7.6 (Baire Category Theorem)**.** *A topological space X is a Baire space if and only each nonempty open set is of second category in X.*

Proof. Suppose X is a Baire space and O is a nonempty open set. Let $\{S_n\}$ be a sequence of sets which is nowhere dense in X and let $O_n = X \setminus \overline{S_n}$. Each O_n is a dense open set of X and, since X is a Baire space, $\bigcap O_n$ is dense. In particular O intersects $\bigcap O_n$ which means that O contains a point not in $\bigcup(X \setminus O_n) = \bigcup \overline{S_n}$. This means O contains a point not in $\bigcup S_n$, so $O \neq \bigcup S_n$ and so O is of second category in X.

Conversely, if X is not a Baire space, then there is a sequence $\{O_n\}$ of dense open sets whose intersection is not dense. Hence, there exists a nonempty open set O such that O does not intersect $\bigcap O_n$. Let $S_n = O \setminus O_n$ and note that each S_n is nowhere dense since $S_n \subset X \setminus O_n$. Also, $O = O \setminus \bigcap O_n = \bigcup(O \setminus O_n) = \bigcup S_n$ which shows O is of first category in X. □

Definition 7.7 (Banach Space)**.** A space X is a Banach Space if it a complete normed vector space.

Theorem 7.8 (Open Mapping Theorem)**.** *Let X and Y be Banach spaces. If $T : X \to Y$ is a continuous linear surjection, then T is an open map.*

Proof. It suffices to show that T maps any open neighborhood of 0 in X to an open neighborhood of 0 in Y. Indeed, if this is shown true and G is any open subset of X, then for every $x \in G$ there is an $r_x > 0$ such that $B(x, r_x) \subset G$. Since $0 \in int(T(B(r_x))$, we have $T(x) \in int(T(B(x, r_x))$ by translation. Thus there is an $s_x > 0$ such that

$$U_x = \{y \in Y : ||y - T(x)|| < s_x\} \subset T(B(x, r_x)).$$

Therefore $T(G) \supset \bigcup\{U_x : x \in G\}$. But $T(x) \in U_x$, so $T(G) = \bigcup\{U_x : x \in G\}$ and hence $T(G)$ is open.

We first show that 0 is in the the interior of the closure of $T(B(r))$ where $B(r) = \{x \in X : ||x|| < r\}$ denotes the open ball of radius r centered at 0. Since T is onto,

$$Y = \bigcup_{k=1}^{\infty} \overline{T\left(B\left(\frac{kr}{2}\right)\right)} = \bigcup_{k=1}^{\infty} k\overline{T\left(B\left(\frac{r}{2}\right)\right)}.$$

By the Baire Category Theorem, there is a $k \geq 1$ such that $k\overline{T\left(B\left(\frac{r}{2}\right)\right)}$ has nonempty interior. Multiplying by $\frac{1}{k}$, we get that $V = int(\overline{T(B(\frac{r}{2}))}) \neq \emptyset$. If $y_0 \in V$, there exists an $s > 0$ such that $\{y \in Y : ||y - y_0|| < s\} \subset V \subset \overline{T\left(B\left(\frac{r}{2}\right)\right)}$. Now let $y \in Y$ be any vector such that $||y|| < s$. Since $y_0 \in \overline{T\left(B\left(\frac{r}{2}\right)\right)}$, there is a sequence $\{x_n\}$ in $B(r/2)$ such that $T(x_n) \to y_0$. There is also a sequence $\{z_n\}$ in $B(r/2)$ such that $T(z_n) \to y_0 + y$. Thus $T(z_n - x_n) \to y$ and $\{z_n - x_n\}_{n=1}^{\infty} \subset B(r)$. Thus $\{y \in Y : ||y|| < s\} \subset \overline{T\left(B\left(r\right)\right)}$ which shows that 0 is in the closure of $T(B(r))$.

Next we show that

$$\overline{T\left(B\left(\frac{r}{2}\right)\right)} \subset T(B(r))$$

as this will show that $0 \in int(T(B(r)))$ for any $r > 0$ and establish T maps an open neighborhood of 0 to an open neighborhood of 0. To that end, fix $y_1 \in \overline{T\left(B\left(\frac{r}{2}\right)\right)}$. By the above, $0 \in int\left(\overline{T\left(B\left(\frac{r}{2^2}\right)\right)}\right)$. Hence $\left[y_1 - \overline{T\left(B\left(\frac{r}{2^2}\right)\right)}\right] \bigcap T(B(r/2)) \neq \emptyset$. Let $x_1 \in B(r/2)$ be such that $T(x_1) \in \left[y_1 - \overline{T\left(B\left(\frac{r}{2^2}\right)\right)}\right]$. Note $T(x_1) = y_1 - y_2$ where $y_2 \in \overline{T\left(B\left(\frac{r}{2^2}\right)\right)}$. Using induction, we obtain a sequence $\{x_n\}$ in X and a sequence $\{y_n\}$ in Y such that:

1. $x_n \in B(2^{-n}r)$;
2. $y_n \in \overline{T}(B(2^{-n}r))$;

3. $y_{n+1} = y_n - T(x_n)$.

Since $||x_n|| < 2^{-n}r$, $\sum_{n=1}^{\infty} ||x_n|| < \infty$. Hence $x = \sum_{n=1}^{\infty} x_n$ exists in X and $||x|| < r$. Also

$$\sum_{k=1}^{n} T(x_k) = \sum_{k=1}^{n}(y_k - y_{k+1}) = y_1 - y_{n+1}.$$

By (3) above $||y_n|| \leq ||T||2^{-n}r$ which implies $y_n \to 0$. Therefore

$$y_1 = \sum_{k=1}^{\infty} T(x_k) = T(x) \in T(B(r))$$

establishing

$$\overline{T\left(B\left(\frac{r}{2}\right)\right)} \subset T(B(r))$$

and completing the proof of the theorem. □

Theorem 7.9 (Inverse Mapping Theorem)**.** *If X and Y are Banach spaces and $T : X \to Y$ is a bounded linear operator that is bijective, then T^{-1} is bounded.*

Proof. By the open mapping theorem T is open and hence a homeomorphism. □

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